Financial Risk Management

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Contents

1 Basic concepts

1.1 Interest rates.

1.1.1 Notation

Definition 1. *$R_{t,T}$ is the per annum interest rate on a deposit from time t until time T.*

For each interest rate the compounding frequency must be determined !

> Example: As an example, let today's interest rate for a 10-year period be $R_{0,10} = 10\%$. If the interest is annually compounded, then, after one year, you get *interest on your interest* (i.e. compounded interest), so if you deposit an amount N_0 today then after one year your amount has grown to (with annual compounding) $N_0(1+R_{0,10})$. Because the interest is (annually) compounded you will, in the second year, receive interest on the interest so after two year the amount has grown to $N_0(1+R_{0,10})^2$. ...
>
> If your interest is compounded *monhtly* then your amount grows, after one month, to $N_0(1+\frac{R_{0,10}}{12})$ and as it is compounded, you it will grow to $N_0(1+\frac{R_{0,10}}{12})^2$ after two months, ... and to $N_0(1+\frac{R_{0,10}}{12})^{12}$ after 12 months, ...

If we generalise this, and denote the compounding frequency as m subperiods of a year, then we find that the interest earned between t and T is given by:

$$N_T = N_t \left(1 + \frac{R^m_{t,T}}{m}\right)^{m\Delta t}, \Delta t = T - t$$

where $R^m_{t,T}$ is the per annum interest rate at time t for for a T-year deposit (starting at t, so from t to $t + \Delta t$) and with a compounding frequency of m per year.

Remark 1.1. *It is important to note that:*

- *T is expressed in years, so one month is 1/12;*
- *m is the number of periods per year;*
- *$R^m_{t,T}$ is per annum.*

If we let $m \to +\infty$ we name it *continuous compounding* and we have[1]:

$$N_T = N_t e^{R^c_{t,T}\Delta t}, \Delta t = (T - t)$$

Note also that, if n is very small, then $(1+x)^n \approx 1 + nx$ so if $m\Delta t$ is small then we have $\left(1 + \frac{R^m_{t,T}}{m}\right)^{m\Delta t} \approx 1 + \frac{R^m_{t,T}}{m} m\Delta t$. For small Δt i.e. for $\Delta t < 1$ we work with $m = 1$ so we have:

[1] because $\lim_{m\to+\infty}\left(1+\frac{x}{m}\right)^m = e^x$

$$N_T = N_t(1 + R^1_{t,T}\Delta t), \Delta t = (T - t)$$

Property 1.

$$N_T = N_t \cdot \Gamma^m_{t,T}, \Delta t = (T - t) \tag{1}$$

where

- *For continuous compounding* $\Gamma^m_{t,T} = e^{R^c_{t,T}\Delta t}$
- *For discrete compounding* $\Gamma^m_{t,T} = \left(1 + \frac{R^m_{t,T}}{m}\right)^{m\Delta t}$
- *For simple annual compounding* $\Gamma^m_{t,T} = \left(1 + R^m_{t,T}\Delta t\right)$

1.1.2 Conversion formulas for compounding.

For continuous compounding, a 1 euro amount grows to $1 \times e^{R^c_{t,T}\Delta t}$ after a time Δt.

For compounding with m subperiods we have after Δt an amount of $\left(1 + \frac{R^m_{t,T}}{m}\right)^{m\Delta t}$.

These yield the same amount if $e^{R^c_{t,T}\Delta t} = \left(1 + \frac{R^m_{t,T}}{m}\right)^{m\Delta t} \iff e^{R^c_{t,T}} = \left(1 + \frac{R^m_{t,T}}{m}\right)^{m}$

Property 2. *From this it follows that the continuous compounding rate with the same yield as the* $m-$*subperiod compounding rate is given by:*

$$R^c_{t,T} = ln\left(\left(1 + \frac{R^m_{t,T}}{m}\right)^m\right) = m \times ln\left(\left(1 + \frac{R^m_{t,T}}{m}\right)\right)$$

Moreover, it follows that $e^{\frac{R^c_{t,T}}{m}} = \left(1 + \frac{R^m_{t,T}}{m}\right)$, therefore

Property 3. *The m-subsperiod compounding rate that has the same yield as for continuous compounding is given by:*

$$R^m_{t,T} = m\left(e^{\frac{R^c_{t,T}}{m}} - 1\right)$$

If we want to convert from an m_1-superiod compounding frequency to an m_2-subperiod compounding frequency (e.g. $m_1 = 52, m_2 = 12$), then in a similar way the convesrion follows from $\left(1 + \frac{R^{m_1}_{t,T}}{m_1}\right)^{m_1\Delta t} = \left(1 + \frac{R^{m_2}_{t,T}}{m_2}\right)^{m_2\Delta t}$ or $\left(1 + \frac{R^{m_1}_{t,T}}{m_1}\right)^{m_1} = \left(1 + \frac{R^{m_2}_{t,T}}{m_2}\right)^{m_2}$

Property 4. *The* m_1*-subsperiod compounding rate that has the same yield as for* m_2*-subperiod compounding is given by:*

$$R^{m_1}_{t,T} = m_1\left(\left(1 + \frac{R^{m_2}_{t,T}}{m_2}\right)^{\frac{m_2}{m_1}} - 1\right)$$

1.1.3 Daycount conventions.

If the time periode Δt is small and in days, then there are several "conventions". Note that time is always expressed in years, but a year can have 365 or 366 days, some months have 28 (29) days, some 30, some 31. This "variability" lies at the source of the so-called daycount conventions to express T or Δt in years:

- Actual/360; count the actual number of days in Δt and take 360 days for a whole year. Note that this can result in e.g. $365/360 > 1$;
- 30/360; assumes that one month counts 30 days and one year counts 360 days;
- Actual/actual; based on the real number of days in a month and the real number of days in a year;

1.2 Discount factors.

The discount factors are derived from these formulas, in fact, the above formulas learn how an amount N_t grows between t and T.

Definition 2. *A discount factor is by definition the inverse operation: it gives the value of an amount N_T at an earler moment $t < T$ or it is the value at t expressed as a percentage of the value at T.*

Therefore the discount factor is the value $DF^m(t;T)$ such that $N_t = DF^m(t;T) \cdot N_T$.

So we find that:

$$DF^m(t;T) = \frac{1}{\Gamma^m(t;T)} \tag{2}$$

Obviously, the discount factor also depends on the discounting frequency.

This implies that the formulas 1 for interest rate compouding can be rewritten in terms of discount factors:

$$N_T = N_t \frac{1}{DF^m(t;T)} \tag{3}$$

1.3 Forward (interest) rates.

$R_{t,T}$ was the (annual) interest rate at time t for depositing money for T years. If t is in the future then this is called a forward rate. So a forward rate is an (annual) interest rate for a period starting in the future and lasting until T. As before, the compounding frequency must also be defined.

Definition 3. *The forward rate at from t until T, $f_{t,T}$ is by definition $R_{t,T}$*

$$f^m_{t,T} \stackrel{def}{=} R^m_{t,T}, \text{ for } t > 0 \tag{4}$$

Example: As an example; we want to know today (2016) an interest rate for a deposit starting in 2020 and ending in 2025.

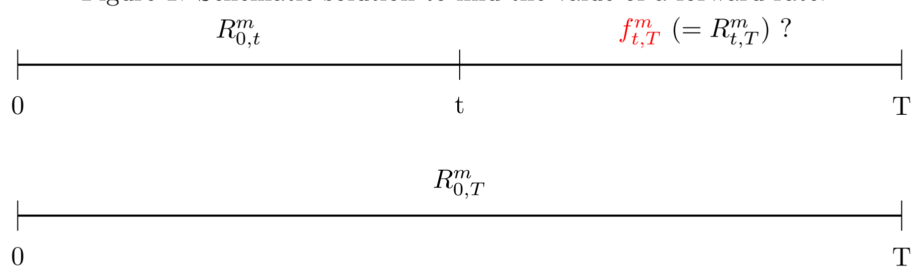
The forward rates can be derived from the interest rates known today, i.e. at $t = 0$. This is in fact rather easy.

If we deposit an amount N_t between t for a time T then it grows to $N_T = \Gamma^m_{t,T} N_t$ (see formula 1) where:

- For continuous compounding $\Gamma^m_{t,T} = e^{R^c_{t,T}(T-t)}$
- For discrete compounding $\Gamma^m_{t,T} = \left(1 + \frac{R^m_{t,T}}{m}\right)^{m(T-t)}$
- For simple annual compounding $\Gamma^s_{t,T} = \left(1 + R^{(1)}_{t,T}(T-t)\right)$

If I deposit, today an amount N_0 until T then I have two options that are schematised in figure 1:

Figure 1: Schematic solution to find the value of a forward rate.



- I deposit it at the rate $R^m_{0,T}$, at the end we will have $N^{(1)}_T = N_0 \Gamma^m_{0,T}$
- I first deposit it at $R_{0,t}$ during a time t and then during $T - t$ at the unknown forward rate $f^m_{t,T} = R^m_{t,T-t}$. Then the amount grows to $N^{(2)}_T = N_0 \Gamma^m_{0,t} \Gamma^m_{t,T}$

Both should be equal, because if one is larger you will of course all choose that and that will change the demand and thus the ineterst rates. So arbitrage makes both the same.

Therefore $\Gamma^m_{0,T} = \Gamma^m_{0,t} \Gamma^m_{t,T}$.

Remark 1.2. *Note that we assumed here that there are "free lunches" are eaten and in "equilibrium" there are no riskless profits, i.e. we assume abscence of risk free arbitrage opportunities.*

Property 5. *In order to find a foward rate, one should follow the procedure schematised in figure 1, i.e. compute the accrued interest over $[0; T]$ in two different ways.*

It follows that $\Gamma^m_{0,T} = \Gamma^m_{0,t} \times \Gamma^m_{t,T}$. *In words this means that the increase over* $[0;T]$ *is the product of the increases over* $[0;t]$ *and* $[t;T]$.

As the $\Gamma^m_{0,t}$ *depend on* $R^m_{t,T} = f^m_{t,T}$ *we can derive* $f^m_{t,T}$.

Remark 1.3. *Note that the foward rates depend on the compounding frequency.*

Replacing the Γ *in* $\Gamma^m_{0,T} = \Gamma^m_{0,t} \times \Gamma^m_{t,T-t}$ *by the definitions below 1 one finds:*

- *continuous compounding:* $e^{R^c_{0,T}T} = e^{R^c_{0,t}t} \times e^{R^c_{t,T}(T-t)}$ *from which you find that* $R^c_{0,T}T = R^c_{0,t}t + f^c_{t,T}(T-t)$, *from which it is easy to find* $f^c_{t,T}$.
- *for discrete compounding:* $\left(1 + \frac{R^m_{0,T}}{m}\right)^{mT} = \left(1 + \frac{R^m_{0,t}}{m}\right)^{m \cdot t} \times \left(1 + \frac{R^m_{t,T}}{m}\right)^{m(T-t)}$
- *for simple compounding:* $(1 + R^s_{0,T}T) = (1 + R^s_{0,t}t) \times (1 + f^s_{t,T}(T-t))$

Remark 1.4. *You do not have to learn these formulas by heart, just remember the procedure schematised in figure 1 and use the appropriate compounding frequency to find the equation. Then solve the equation for* $f^m_{t,T} = R^m_{t,T}$.

1.4 The term structure or the yield curve.

1.4.1 Definition

Definition 4. *A zero (coupon) rate for maturity* T *is the rate of interest earned on an investment with a payoff only at* T. *So there are no coupon payments (or intermediate interest payments) between* t *and* T.

One can compute a zero rate for every maturity T, i.e. for T being, one week ,one month, ...

Definition 5. *The term structure or the yield curve is the function that maps* T *to the zero rate for maturity* T. *Graphically it has* T *on the horizontal axis and the zero rate corresponding to* T *on the vertical axis, i.e. in the graph you put the couples* $(T, R_{0,T})$ *for different values of* T.

Definition 6. *A 'normal' bond, contrary to a zero coupon bond, pays interests periodically, e.g. you get the interest every six months.*

A bond has several parameters:

- *A principal or a face value; this is the amount the interest is computed on;*
- *A maturity; this is the date at which the principal is paid to the owner of the bond;*
- *An interest rate (including the compounding frequency);*
- *The periodicity of the interest payments.*

1.4.2 Bootstrap forward rates.

The yield curve can be expressed in two ways:

- in terms of the zero coupon rates: so at T on the horizontal axis you find $R_{0,T}$ on the vertical axis, so graphically you plot $(T, R_{0,T})$ for different values of T
- more unusual but also used is in terms for forward rates; for T on the horizontal axis you find $f_{T-\Delta t,T}$ on the vertical axis, so graphically you plot $(T, f_{T-\Delta t,T})$ for different values of T.

Both can easily be transformed to one another via the formulas supra.

Remark 1.5. *Note that, using market data and bootstrapping, we can find several pounts of the graph, i.e. $(T, R^c_{0,T})$, for different values of T.*

In that case we get a scatterplot, we could fit a line through these points like e.g. a linear regresstion of the type $R^c_{0,T} = \beta_0 + \beta_1 T + \epsilon$. However, a linar shape does not seem to fit quite well, therefore a non-linear regression is more appropriate. One often used function is the one proposed by Nelson-Siegel. In stead of a linear shape they propose $R^c_{0,T} = (\alpha_1 + \alpha_2 T)e^{-\alpha_3 T} + \alpha_4 + \epsilon$.

Using bootstrapping we can find several observations $(T, R^c_{0,T})$, for different values of T, and using these data points and regression techniques we can find estimates of $\hat\alpha_1, \hat\alpha_2, \hat\alpha_3, \hat\alpha_4$. The equation for the yield curve, with which one can compute the rates for intermediate time points, is then given as

$$R^c_{0,t} = (\hat\alpha_1 + \hat\alpha_2 t)e^{-\hat\alpha_3 t} + \hat\alpha_4, \quad \textit{(Nelson-Siegel)}$$

Note that,

- *for $t = 0$, i.e. the very short term zero rate, it is $R^c_{0,0} = (\hat\alpha_1 + \hat\alpha_2 0)e^{-\hat\alpha_3 0} + \hat\alpha_4 = \hat\alpha_1 + \hat\alpha_4$;*
- *for $t = +\infty$, i.e. the very long term zero rate, it is $R^c_{0,+\infty} = \hat\alpha_4$*

see exercise 11.1.5

1.4.3 Duration of bonds.

Assume you have an asset with cash flows c_t at dates $t = 1, 2, \ldots, T$. This might be a bond (in which case c_t are interests paid for $t < T$ and c_T is interest and principal amount) or a share of equity (in whic case the c_t are dividends). Note that some of the c_t may be zero and the do not have to be equal.

Taking the zero coupon bond with maturity T (see section 1.4) as a special case, the time you have to weight to receive cash payments is T because you receive all the money at maturity (this is the defintion of a zero coupon bond). We say that the *duration* of the bond is T years.

If you have a "normal" bond then you will receive periodic payments between 0 and T, and at T you also receive the face value. Obviously, you do not have to wait until T to receive cash, so the duration of a (coupon bearing) bond is less than T.

Definition 7. *The duration of a bond is the average time (in years) you have to wait until you receive cash payments.*

If we discount at rate y then the present value of the series of cash flows (interest payments and interest+face value at T) is the price of the bond: $B = \sum_{t=1}^{T} c_t e^{-y\times t}$, t in years.

$$B = \sum_{t=1}^{T} c_t e^{-y\times t} \tag{5}$$

For a zero coupon bond (only payments at T) this becomes $B = c_T e^{-y\times T}$ where c_T is the sum of interest and face value. If we define the duration as $T \times \frac{c_T e^{-y\times T}}{B}$ then this will be T. Therefore we define duration as:

Definition 8. *The duration of a bond is the weighted sum of the time values where you recieve cash, the weights are the fractions of the total present value paid, wo* $w_t = \frac{c_t e^{-y\times t}}{B}$ *and the duration is*

$$D = \sum_{t=1}^{T} w_t \times t = \sum_{t=1}^{T} \frac{c_t e^{-y\times t}}{B} \times t \tag{6}$$

Note that, by the above, it holds that for a zero coupon bond $D = T$ because the weights $w_t = 0$ for $t < T$ and at T the weight is $\frac{c_t e^{-y\times T}}{B} = \frac{B}{B}$, so $D = 0 + 0 + \cdots + T = T$.

We can look how the bond value changes with the rate, i.e. $\frac{dB}{dy}$. Using equation 5 we find $\frac{dB}{dy} = -\sum_{i=1}^{T} t \cdot c_t \cdot e^{-y\times t}$ or $dB = -dy \sum_{i=1}^{T} t \cdot c_t \cdot e^{-y\times t}$.

From equation 6 we find that $\sum_{i=1}^{T} t \cdot c_t \cdot e^{-y\times t} = B \cdot D$ such that:

Property 6. *The percentage change in the bond value* $\frac{\Delta B}{B} \approx \frac{dB}{B}$ *is given by*

$$\frac{dB}{B} = -dy\dot{D} \tag{7}$$

or percentage change in the bond value is equal to the rate change times the duration (with a minus sign).

So the duration is a measure of sensitity of the bond value to interest rate changes. This explains why the duration has become such a popular measure[2]

[2]Duration was invented by Macaulay in 1938).

2 Forwards and futures

2.1 Definition of a derivative

Definition 9. *A derivative (or contingent claim) is an instrument whose value depends on the value of another, underlying variable.*

Examples are forwards, futures, options, swaps, ...

Remark 2.1. *The underlying asset can be*

- *Traded: examples are*
  - *currencies;*
  - *equity;*
  - *gold;*
  - *...*
- *Non-traded: examples are*
  - *interest rates;*
  - *credit quality of a company;*
  - *temparature;*
  - *inflation;*
  - *...*

Remark 2.2. *In "pricing" derivatives their fair value will often be determined using arbitrage. Therefore it is important to keep in mind the distinction between consumption assets and investment assets.*

Consumption assets are held mainly for consumption (i.e. short-term) while investment assets are held for investment purposes (long-term).

seldomly works for consumption assets.

Remark 2.3. *Derivatives can be used for several purposes; hedge risk, speculation, aribtrage, change the nature of an asset, ...*

2.2 Definition of a forward contract.

Definition 10. *A forward contract is an agreement between a seller and a buyer to sell or buy an asset at a certain time in the future for a certain price.*

The parameters of a forward contract are:

- *an* ***obligation*** *to buy or sell an asset (e.g. gold) in some* ***currency*** *(e.g. euro);*
- *a* ***quantity*** Q *of an* ***underlying asset*** *in* ***units of the asset****;*
- *at a specified time in the future, i.e. with a* ***time to maturity*** T

- *at a certain* ***delivery price (or contract price or excercise price)*** *K (or X)*

obligation to sell/buy: *The party that is obliged to sell at the future time period is said to have a* ***short forward position****, the one that is obliged to buy in the future is said to have a* ***long forward position****.*

Remark 2.4. *A forward is just an agreement between the seller and the buyer, so writing the contract does not cost anything.*

2.3 Delivery price of the forward contract.

2.3.1 Delivery price derivation - arbitrage

The price that is agreed to be paid in the future is called the delivery price of the forward contract or short the *forward price*. Note that this is the price that is in the contract. We will see that the contract itself also can have a (fair) value, so it is important to distinguish the delivery price in the contract from the price of the contract.

The "forward price" (i.e. the price in the contract) is a price for future delivery, so we need to find out a way to fix that price in a "reasonable" way. These prices are derived using *"arbitrage arguments"*. Let's see how it works for a forward price:

"Arbitrage" means that we try to find riskless positions that are profitable. So let's analyse the following combined position:

1. We write a future where we agree to sell a forward at someone else, the delivery (unit) price in the contract is K. We also have to fix values for T and Q and the underlying asset in the contract;

   The cost of writing such a contract is zero of course.

2. We borrow money at the bank to buy an amound Q of the asset on the spot market. The price in the spot market today is of course known and let it be equal to S_0, so we need to borrow QS_0 and use this money to buy Q amount of the asset on the spot market.

   Note: it is assumed here that we can always buy the asset on the spot market.

After a time T this combined position yields:

1. I agreed to sell the asset at a price K, so at T we receive $Q \times K$;

   Note: it is assumed here that taxes have no impact.

   Note: it is assumed here that there are no transaction costs.

2. after T we have to pay back the borrowed money and the interest on it, *assuming* continuous compounding, this means that, at T we have to pay $Q \times S_0 e^{rT}$ to the bank;

   Note: it is assumed here that one can borrow at the risk free rate.

Note: if the possession of the asset would imply any profits or costs (e.g. with consumption assets), then these have to be included in the reasoning. If i_T are the total profits between zero and T, valued at T, and similar for c_T then this becomes $Q \times S_0 e^{rT} + c_T^{[0;T]} - i_T^{[0;T]}$, where $i_T^{[0;T]}, c_T^{[0;T]}$ have different forms depending in the type of income/costs included in i, c. Note that i_T, c_T are valued at T. If we formulate it in terms of discounted values to $t = 0$ then this becomes: $Q \times \left(S_0 + c_0^{[0;T]} - i_0^{[0;T]}\right) e^{rT}$

3. Therefore, at T we receive $Q \times K$ and have to pay $Q \times S_0 e^{rT}$.

If $Q \times K$ would be larger than $Q \times S_0 e^{rT}$ then this a (positive) profit and nowhere did I take any risk. Of course, if things are so easy then I will write more forwards like that and borrow more money at the bank and buy at the spot, so, because of an increased demand, this will increase r just as well as S_0 and both amounts come closer to each other. If $Q \times K$ would be smaller than $Q \times S_0 e^{rT}$ then I will loose at T, so I will not write forwards, and thus not borrow and buy at the spot, therefore r and S_0 will decrease !

So we find that, after arbitrage, the "fair value" for the forward price that is in the contract and that is decided at $t = 0$ is $K = F_0 = S_0 e^{rT}$, where S_0 is the spot price at $t = 0$ and r is the *risk free* interest rate. Risk free because the position we have taken in entail no risk, so the bank will grant us the risk free rate for that loan.

Note: it is assumed here that risk free arbitrage opportunities do not exist.

The fair value for the delivery price in the forward contract is given by:

$$F_0 = \left(S_0 + c_0^{[0;T]} - i_0^{[0;T]}\right) e^{r_0 T}$$

We will make some elements more explicit, because that will seem to be of big importance for currency forwards:

Definition 11. *The fair value for the delivery price in the forward contract is given by:*

$$F_0^{EUR/Asset} = \left(S_0^{EUR/Asset} + c_{0,[0;T]}^{EUR/Asset} - i_{0,[0;T]}^{EUR/Asset}\right) e^{r_0 T} \quad (8)$$

where

- $F_0^{EUR/Asset}$ *is the price in the contract, i.e. the price agreed to pay at the future period T;*
- $S_0^{EUR/Asset}$ *is the spot market for the asset today;*
- r_0 *is the risk free rate today;*
- *we assumed continuous compounding ($e^{r_0 T}$);*

- $c_{0,[0;T]}^{EUR/Asset}$ *are the costs from keeping the asset in* $[0, T]$*, costs valued at* $t = 0$ *(so the type of compounding is important);*

- $i_{0,[0;T]}^{EUR/Asset}$ *are the profits from keeping the asset in* $[0, T]$*, costs valued at* $t = 0$ *(so the type of compounding is important);*

Remark 2.5. *Note that, the way we derived the formula, if we sign at* $t = 0$ *a forward and we take for the delivery price* X *this value of* F_0*, then there is no profit nor a loss at* $t = 0$*, i.e. if the prive in the contract is* $X = F_0$ *defined supra, then the value of the contract at* $t = 0$ *is zero.*

Remark 2.6. *We have different terms and factors where compounding interest rates is required. be very carefull which one to use, they may even be different !!!!*

E.g. for the income, it could be that they are $\Delta t < 0$ *so with simple compounding, while for* $e^{r_0 T}$*, the* T *may be far in the future, so certainly not simple annual compounding !*

2.3.2 Delivery price when the asset has a profit that is a percentage of its value.

In the previous section we saw that assets the delivery price depends on profits earned between [0;T]. A special case of profits arise when the profit is a percentage of the asset value and when the profit is compounded. So let us assume an asset like that.

As in the previous section we compare two alternative strategies:

1. Write a forward to sell an amount Q at a price X at T;
2. Borrow money and buy at the spotmarket for a spot price of S_0;

The second alternative requires you to pay back the loan at T; $S_0 Q e^{rT}$ as before. However, we have bought Q units of the asset and each unit now earns a percentage q of its value, so at the end we have Qe^{qT} units of the asset.

- Continuous compounding:

  Therefore the Qe^{qT} units of the asset cost us at T an amount of $S_0 Q e^{rT}$, or at T a unit of the asset will cost $\frac{S_0 Q e^{rT}}{Q e^{qT}} = S_0 e^{rT - qT} = S_0 e^{(r-q)T}$.

- In the case of discrete compounding,

  with a similar reasoning, we will find e a cost of $Q \times S_0(1 + \frac{r}{m})^{mT}$ for $Q(1 + \frac{q}{m})^{mT}$ units or a unit cost of $S_0 \frac{(1+\frac{r}{m})^{mT}}{(1+\frac{q}{m})^{mT}}$

- With simple annual comounding

  with a similar reasoning, we will find e a cost of $Q \times S_0(1+rT)$ for $Q(1+qT)$ units or a unit cost of $S_0 \frac{(1+rT)}{(1+qT)}$

So, for an asset that yields profits that are a **percentage of their value and that are compounded** the delivery price will be:

- Continuous compounding;

$$F_0^{c,EUR/Asset} = S_0^{EUR/Asset} e^{(r_0 - q_0)T}$$

- Discrete compounding;

$$F_0^{c,EUR/Asset} = S_0^{EUR/Asset} \frac{(1+\frac{r}{m})^{mT}}{(1+\frac{q}{m})^{mT}}$$

- Simple annual compounding:

$$F_0^{c,EUR/Asset} = S_0^{EUR/Asset} \frac{(1+rT)}{(1+qT)}$$

2.4 The (fair) value of a forward contract.

2.4.1 Value of a forward contract at maturity T

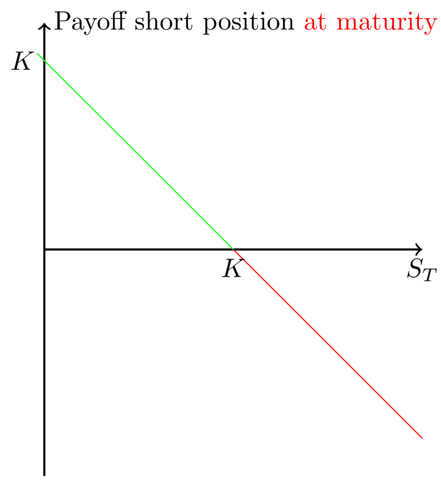
If, at $t = 0$ we sign a forward contract with parameters K, Q, T and the asset, then after a time T we can compare the delivery price in the contract to the spot market price a T, i.e. S_T:

- **If you are the seller** in the contract, then, if
  - $S_T > K$, then you have to sell at K but you could have sold it at the spot for $S_T > K$, so you have lost money, or your profit is negative, namely $K - S_T < 0$;
  - $S_T < K$, then you can sell at K while at the spot you would have had to sell at $S_T < K$, so you win $K - S_T > 0$

  Summary: For the seller, i.e. the one with the short position in the forward contract, the profit at T, valued at T, is $Profit_T = K - S_T$, for every value of S_T;

  Graphically, in a graph $(S_T, Profit_T)$[3] this is a line with a negative slope, the higher S_T, the more you "loose". This is shown below, where the line is red there is a loss, where it is green there is a profit.

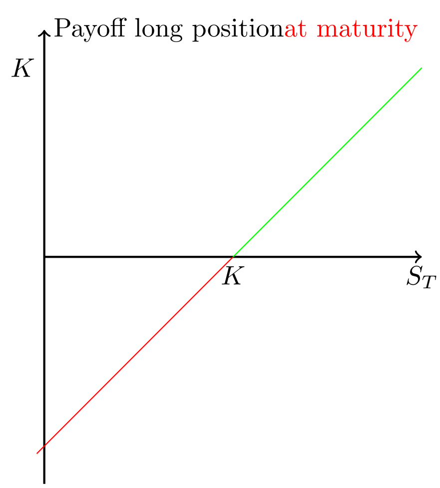
[3]This is S_T on the horizontal axis and $Profit_T$ on the vertical one.



- **If you are the buyer** in the contract, then clearly the position is opposite:

  Summary: For the buyer, i.e. the one with the long position in the forward contract, the profit at T, valued at T is $Profit_T = S_T - K$, for every value of S_T;

  Graphically, in a graph $(S_T, Profit_T)$ this is a line with a positive slope, , the higher S_T, the more you "earn".



2.4.2 No physical delivery.

At T, two things can be done:

1. The seller buys the asset on the spot at S_T and then supplies the asset to the buyer and the buyer pays K per unit (at T);

2. The seller pays an amount $S_T - K$ to the buyer to buy back the forward contract, the buyer uses K of his own money and using this K and the $S_T - K$ of the seller, he can buy at S_T on the spot market.

The latter is what mostly happens, the contracts are bought back before the expiration date and physical delivery seldomly takes place ! So the market for deliverables is a "paper market".

This makes it attractive for speculators; indeed, they do not need any assets, they just have to write a contract (but as seen in the course, they need margins).

2.4.3 Value of a forward contract at a time $t, 0 < t < T$

We saw in section 2.4.1 that, when there is a "mismatch" between the contract price K and the spot price at T, S_T (which is unknown when the contract is signed) the contract itself has a (fair) value.

If the value is not zero at T, then it will obviously also have a value at t, $0 < t < T$. We denote that value as f_t.

Section 2.4.1 learned that, **for a long position**, at T it holds that $f_T = S_T - K$. At $t < T$ this value has to be discountd with the rate at t, it is therefore at t equal to $f_T e^{-r_t(T-t)} = (S_T - K)e^{-r_t(T-t)}$.

Of course, at $t < T$ the value of S_T is not known, we only can know the fair value namely $(S_t + c_t^{[t;T]} - i_t^{[t;T]})e^{r_t(T-t)}$.

So we find that $f_t = \left((S_t + c_t^{[t;T]} - i_t^{[t;T]})e^{r_t(T-t)} - K\right) e^{-r_t(T-t)}$.

If K is (at $t = 0$) choosen at its fair value given in 8, then we find that **for a long forward position** and with **continuous compounding**:

$$f_t^c = (F_t^c - F_0^c)e^{-r_t(T-t)} \text{ (Long position)} \tag{9}$$

where F_t^c and F_0^c are both computed using the formulas:

$$F_t^c = (S_t + c_t^{[t;T]} - i_t^{[t;T]})e^{r_t(T-t)} \tag{10}$$

$$F_0^c = (S_0 + c_0^{[0;T]} - i_0^{[0;T]})e^{r_0(T-0)} \tag{11}$$

So we find that,
If K is (at $t = 0$) choosen at its fair value then $f_0 = 0$:

- If the delivery price in the contract is chosen at its fair value given in 8, then the contract value at the start is zero;
- In other words, the fair value for the delivery price in the contract is that value of the delivery price that makes the value of the contract zero at $t = 0$.

Similar formulas can be found with discrete compounding:

$$f_t^m = (F_t^m - F_0^m)(1 + \frac{r_t}{m})^{-m(T-t)} \text{ (Long)} \tag{12}$$

where F_t^m and F_0^m are computed with discrete compounding.

Remark 2.7. *To already make an analogy with option valuation that will come later, we can also find the value of the contract using a "replicating" portfolio.*

Assume we are at t an we make a portfolio of (a) short position in a future with exercise price X and maturity T and (b) borrow money to buy the underlying on the spot market. Then at T we have to pay back the loan or $-S_t e^{r(T-t)}$ and we receive X because of the obligation to sell in the short forward position. Therefore the value at T of this replicating portfolio is $X - S_t e^{r(T-t)}$ and discounted to t this becomes $(X - S_t e^{r(T-t)})e^{-r(T-t)} = Xe^{-r(T-t)} - S_t$.

This is the same formula as supra because when X is fixed at $t = 0$ it will be set at the F_0 supra.

What is important to note here is that (see options later) the number of stock in the replicating portfolio (later we will call this Δ) is exactly 1. So we find that the Δ of a forward is equal to 1. Thus to make the replicating portfolio risk neutral we have to make a portfolio of one unit of stock and one short position in a forward.

Note that 1 is also the derivative of the forward value.

2.5 Futures versus forwards.

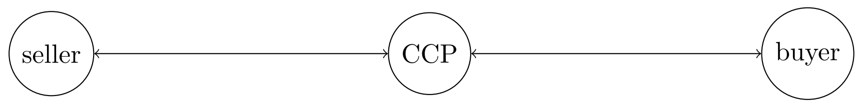
2.5.1 Differences between futures and forwards

A forward contract is a taylor-made contract between a seller and a buyer, they both agree on the underlying and its quality, on the amount, on the forward price, ...

Obviously, if they are taylor made, they fit the requirements of the buyer and the seller, but they are mostly of low interest to other agents. Therefore their liquidity is low, i.e. it is not easy to sell these to someone else.

To increase the liquidty a more standardised type of contract was created, this "standardised forward" is called a future. So the underlying in the future is a well-defined asset of s very specific quality. The amount in the contract and the maturity date are also standardised. This standardisation will make it more easy to sell it to other agents, so standardisation increases the liquidity. Obviously, this comes at a cost; a future is less flexible and if you want to hedge your own asset price, it might be that there is no future contract that perfectly fits your own asset. So the hedge is approximate (this is one of the causes of "basis risk", see section 2.7).

Another distinguishing feature is the parties involved in the contract; a forward is "over-the-couter" (OTC) while a future is traded on an exchange. The OTC contract is an agreement between only two parties, the seller and the buyer. In a future contract the "counterparty" is always the exchange; so if you want to sell a future you sell it to the exchange, therefore the exchange is the Central Counter Party (CCP). Similar if you want to buy, you buy from the exchange.



Obviously, if all these contracts pass through the central counter party, then all the risk is centralised at this central party; if you buy a future from the central party and at T you cannot pay, then the central party has a loss because of your credit risk. If this happens with many customers of the central counterparty , then this central counterparty may go bancrupt. Therefore, everyone participating on the future exchange will have a daily settlement of the profits and the losses. If you buy a future today, and tomorrow the prices have changed, then you have to put a "reserve" called a margin at an account with the central counterparty. This process of adapting the positions daily, using margins, is called marking to market (MTM).

This marking-to-market (or margining) reduces the credit risk taken on by the exchange.

2.5.2 Marking to market; re-write future contract daily.

To illustrate how this "marking-to-market" works in practice let's take an example. We sign a contract with the CCP to sell at T in the future. As we already know, the future price (i.e. the price <u>in</u> the contract) is determined such that , at the moment of signing the contract, the value <u>of</u> that future contract is zero. Obviously we know the spot price of the (standardised) underling today, i.e. S_0 and (in the simplest case) the future price is $X = F_0 = S_0 e^{r_0 T}$

When one day has passed, we know S_1 and F_1 should be $S_1 e^{r_1(T-1)}$. The contract was signed at $t = 0$ so in the contract we have $X = F_0$, where F_0 was fixed "yesterday" as $F_0 = S_0 e^{r_0 T}$. However, the day after the price should be F_1. If F_1 is different from F_0 (and this will usually be the case because S_1 may have changed or r_1 may have changed) then you have a profit or a loss (depending on whether you have a short or long position or whether $F_1 > F_0$ or vice versa). This profit (or loss) is put on your margin account that day (or withdrawn from it if you have a loss), this is exactly the "marking-to-market", and the contract is re-written with a contract price $X = F_1$!.

So in stead of "accumulating" the value over the whole period t, T, you get paid every day and the profit (loss) is paid on (withdrawn from) your account, while at the same time re-setting the value <u>of</u> the contract to zero (because you got the money on your account, so the total must be reset to zero). Setting the value <u>of</u> the contract to zero is the same as re-writing the future price <u>in</u> the contract to $X = F_1$ (see supra).

This daily marking to market does <u>not</u> take place with a forward contract !! For a forward the only exchange of money in a forward is at $t = T$ where (for a long position) you receive $S_T - X = S_T - F0$!

This can be schematised as follows as below; on the left hand side we schematise the marking-to-market for the future, on the right hand side the settlment at T for the forward. Assume that $T = D/365$ (we use days because marking to market is daily)

For a future (left hand side) you make a contract with the Central Counter Party of the exchange. The initial price is, as usual, such that the value of the contract at $t = 0$ is zero, i.e. $F_0 = S_0 e^{r_0 T}$. As you are on the exchange you also have to deposit an initial margin m_0 (which serves just as a guarantee, you get it back if you quit the exchange). For the forward you find another seller (e.g.) and F_0 is identical.

After 1 day the future is "marked-to-market" meaning that the value $F_1 - F_0$ is added to your margin account and the contract is re-written such that its value is zero (because you were paid the value on your margin account), i.e. the price in the contract is re-written to make the value of the contract equal to zero. This is shown in the second line.

For the forward there is ony settlement at $t = T$, so in the first row nothing happens.

The other rows are similar.

t	X (future)	margin (future)	X (forward)
0	$X = F_0$	m_0	$X = F_0$
1/365	$X = F_1$	$m_1 = m_0 + F_1 - F_0$	-
2/365	$X = F_2$	$m_2 = \overbrace{m_1}^{m_0+F_1-F_0} + F_2 - F_1 = m_0 + F_2 - F_0$	-
3/365	$X = F_3$	$m_3 = m_2 + F_3 - F_2 = m_0 + F_3 - F_0$	-
⋮	⋮	⋮	⋮
T	$X = F_T = S_T$	$m_T = m_0 + S_T - F_0$	$S_T - S_0$

After T we can see that our margin account has grown from m_0 to $m_0 + S_T - F_0$, or we received $S_T - F_0$ from the future (this amount could be negative), while for the forward we receive the same amount at T.

2.5.3 Are prices of forwards and futures equal ?

The above schematised situation in the table may be misleading, at $t = T$ we receive the same amount, but could it be that the intermediate payments for the future make us earn (or lose) interest on the intermediate payments ? Could that have consequences for the being equal of prices of forwards and futures ? (empirical results, see slide 41 and next, show that differences may exist.)

Property 7. *If the risk-free rate is constant*[4] *and equal for all maturities*[5] *then forward prices are equal to future prices. This also holds if the risk-free rates are a known function of time.*

This does not hold if the interest rates are stochastic, i.e. uncertain in the future (which is usually the case):

Let us try to show this in the simples case; the risk free rate is constant and the same for all maturities (but the reasoning is the same for a known function of time).

[4] This is the interest rate for a loan of 1 year is the same in 2015, 2016, ...

[5] This is the interest rate for a loan of 1 year in 2015, is the same as the two-year rate in 2015 ...

Let G_t be the future price (in the contract) at t for a future maturing at T and F_t be the forward price (in the contract) at t for a forward maturing at T. We have to show that $G_t = F_t, \forall t$:

It holds at $t = T$: This is obvious since at maturity both prices converge to the spot price at T, i.e. $F_T = S_T$ and $G_T = S_T$ and therefore $F_T = G_T$.

At $t = T - 1$: Build a portfolio where

- you go long on a forward with $X = F_{T-1}$
- you go short on a future with $X = G_{T-1}$

This costs nothing, you just sign contracts !

Then, at T, the profit from the long position is $S_T - X = S_T - F_{T-1}$. Indeed, at T you can buy (long position) the asset for X and sell it in the market for S_T, or you have $S_T - X$.

The profit from the short position is $G_{T-1} - G_T$. Indeed, you have to sell at G_{T-1} and you can buy at G_T, or you get $G_{T-1} - G_T = G_{T-1} - S_T$ (we have already shown that $G_T = S_T$ in our first step).

For the whole portfolio this gives $S_T - F_{T-1} + G_{T-1} - S_T = -F_{T-1} + G_{T-1}$.

At $t = T - 1$ the portfolio costs nothing, so if at $t = T$ this would not value to zero there would be arbitrage opportunities and this will yield a zero profit at T, so $F_{T-1} = G_{T-1}$

At $t = T - 2$: Build a portfolio where

- you go long on n forwards with $X = F_{T-2}$
- you go short on one future with $X = G_{T-2}$

we will try to find a value for n, but it must be possible to compute that at $T - 2$ because if not we do not know how to build that portfolio.

At $T-1$ we close both positions,we have to be carefull here,because for the forward we do not receive anything unitil T ! So at $T - 1$ we must take the value at T and discount it. Therefore, from the long position we have $n(F_{T-1} - F_{T-2})$ but this is received at T so we have to discount to $T - 1$. As interest rates are contant and the same for all maturities (note that we are not at $T - 2$, it may be in the future) we can discount this: $e^{-r\Delta t} n(F_{T-1} - F_{T-2})$, here r is a fixed number.

On the future, the profits are marked-to-market daily, so you do recieve $G_{T-2} - G_{T-1}$.

So at $T - 1$ you have $e^{-r\Delta t} n(F_{T-1} - F_{T-2}) + G_{T-2} - G_{T-1}$ as the value of the portfolio. As this value was orginally zero (we just signed contracts), this should be zero else there are opportunities for arbitrage !

We did not say how we would fix the n, so we can still choose that, as long as n can be computed at $T-2$!. So if we choose $n = e^{r\Delta t}$ then, as r is known and constant, we can compute n at $T-2$. In that case the profit is $e^{-r\Delta t}e^{r\Delta t}(F_{T-1} - F_{T-2}) + G_{T-2} - G_{T-1} = (F_{T-1} - F_{T-2}) + G_{T-2} - G_{T-1}$.
For the same reason as supra $(F_{T-1} - F_{T-2}) + G_{T-2} - G_{T-1}$ should be zero, but we have already proven that $F_{T-1} = G_{T-1}$ so we find that it must be that $F_{T-2} = G_{T-2}$.

At $t = T - n$: similar as for $T-2$.

Remark 2.8. *Note that, in the step $t = T-2$ we "replicated" the future by a portfolio of n forwards to make it riskless. In other words we hedged the future using $n_{T-2} = e^{r_{T-2}(T-(T-2))}$ forwards.*

Note that this n_* can be different because the riskless rate may change in tome (but we have to know it at $t = *$ and because $(T-t)$ changes. So during the life of the future between $[0, T]$ we have dynamically constructed a hedge with forwards.

There is a remark on the Δ of these hedges on slide 39. We already found that the Δ of a forward is 1. We need $n = e^{r\Delta t}$ futures to hedge a future, therefore the Δ of a future is $n = e^{r(T-t)}$ which becomes 1 at $t \to T$.

2.6 Some special cases

2.6.1 Forwards and futures on currencies.

A forward(or future) on a currency is a special case of forward (future), namely

1. one where the asset is itself a money-unit;
2. as the asset is also a currency, interest is earned on it, so it has a profit being a percentage of its value (see section 2.3.2)

Assume that at T you want to buy USD, then, the thing that you want to buy is whatis called the asset, so our asset is $A = USD$. We want to buy it with EUR, so the money we pay with is EUR.

Then today the forward price, i.e. the price in the contract is fixed such that the vaue of the future is zero today. Note however that in this case the asset $A = USD$ has a profit because we earn interest on it. So the q is the profit on the asset, and in this case the asset is USD so q in section 2.3.2 is the risk free rate on the USD, r_{USD}.

The r in section 2.3.2 is the discount factor, which is the interest rate on the meny that you use for paying the asset, so in our case it is r_{EUR}

Now we can just apply the rules we already know, only you should be very cerfull and keep in mind which is the asset (a currency) and which is the mney you pay with. In our example we pay with EUR and the asset is USD ($A = USD$).

Property 8. *So the forward price (i.e. the price in the contract) at t for $0 \leq t \leq T$ with continous compounding is $F_t^c = S_t e^{(r_t - q_t)(T-t)} = S_t^{EUR/A} e^{(r_{EUR,t} - q_{A,t})(T-t)}$ and as the asset A is USD we find:*

$$F_t^{c,(EUR/USD)} = S_t^{(EUR/USD)} e^{(r_{EUR,t} - r_{USD,t})(T-t)}$$

Applying this to $t = 0$ we can find the value of the cotract.

Property 9. *If we now compute the value of the contract with an asset that yields a percentage profit then, as before, we have (note that discounting is always with the money you use to pay:*

$$f_t^{c,(EUR/USD)} = (F_t^{c,(EUR/USD)} - F_0^{c,(EUR/USD)}) e^{-r_{t,EUR}(T-t)}$$

You can learn thse formula by heart, but if you remember the "trick" with the money that is the asset and the money used for paying then it is just an application of what you already know.

2.6.2 Futures on a stock index

Definition 12. *A stock index is a hypothetical portfolio of stocks, i.e. a portfolio with (hypothetical) quantities of shares of several stocks. E.g. in the Bel 20 you have shares of 20 large Belgian companies.*

E.g. a fictituous index could be (50 shares of Electrabel, 70 shares of Imbev). The value of the index at a point in time t is then $50S_{EBel,t} + 70S_{Imbev,t}$. Obvioulsy, because the prices of the shares of EBel and Imbev change through time, the value of the index will hcnage in time.

Property 10. *As the index can be associated to the value of a hypothetical portfolio, it is just a special kind of asset. In other words it can be used as an unerlying in a future (or option). Note however that the shares in the index pay dividends, for an index (this is not the case for every share) the divident yield is a percentage of the value, so the formulas from section 2.3.2 are applicable.*

2.6.3 CAPM and hedging portfolios using stock indices.

We have seen that the value of an index evolves in time, according to its composition and to the prices of the shares in the index.

Of course you can yourself have a portfolio of equity that "looks very much" like the index, i.e. the price of your own portfolio andthe price of the index move more or less equally.

It is evident that you might loose money in the future if the value of your portfolio decrases a lot. So how can you hedge against this risk ?

You have your portfolio that is worth Po_0 EUR today, of course you do not know the value of it a $T, T > 0$ (because you can not foresee the future).

If you wnat to be "sure" abot a price in the future then you can short futures as follows:

- Assume that the price in the futures contract today is F_0, and that each contract is about Q units of the index. Then one future contract is worth $V_{F,0} = Q \times F_0$.
- If you short (i.e. an obligation to sell in the future) $N = \frac{Po_0}{V_{F,0}}$ futures, then you have no more uncertainty about the value of your portfolio at T.

In the example we assumed that the prices of our portfolio and the value of the index evolved in a similar way. Obviously this does not hold for every portflio, however the Capital Asset Pricing Model brings the solution. CAPM allows to estimate the sensitity of your own portfolio compared to the index using the so-called β of your portfolio with respect to the index.

If the β of your portfolio is e.g. 2 then you have to short twice as much futures to hedge your risk. In general you will need $N = \beta \frac{Po_0}{V_{F,0}}$.

If your portflio closly resembles the index then $\beta = 1$ and you find the result supra.

2.6.4 Futures on interest rates.

Definition 13. *An interest rate future is a future contract where the underlying asset is an asset that pays interests, e.g. a bond.*

They can be used to hedge aginst interest rate changes at time periods in the future.

Assume you own a zero coupon bond with an continuously compounded interest rate $r = 5\%$ that matures in 5 year and a principal of 100. Then after 5 year you will receive $100^{0.05 \times 5} = 128.4025417$. The contract was signed at 30/11/2015. You will then receive in 5 year 128.4025417.

Assume that after one year, i.e. on 30/11/2016, the interest rates change to 4%. Then,

- with the original interest rate of 5% your bond is worth 128.4025417 on 30/11/2021, so discounting it to 30/11/2016 at 5% this would be 105.1271096
- However, the interest rate has changed, so you will receive 128.4025417 on 30/11/2021, discounted at 4% is 109.4174284.

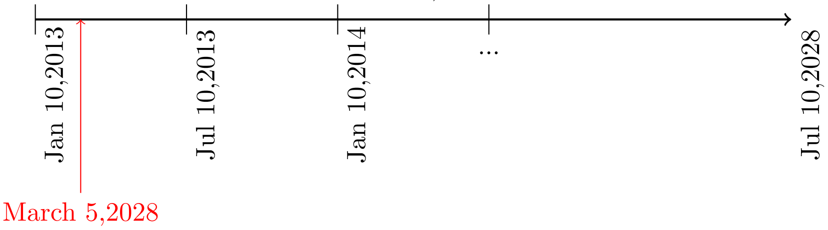
Property 11. *So it follows that, if the interest rates decrease then the value of an existing bond increases, if the rates increase then the value of the bond decreases.*

Obviously, when this bond is used as an underlying asset in a future or a forward, then the future price (in the contract) will also change.

Therefore a future on a bond, which is according to the definition supra an interest rate future, can be used to hedge against changes in interest rates.

The price of a forward was determined buy (a) buying the asset with borrowed money and (b) shorting a fowrad on the asset at the same time. For interest futures there are some particularities.

The first particularity with a bond is that you receive periodic interest payments, e.g. every six months. Assume you have a bond with semi-annual interest payments, a face value of 100 and an interest rate of 11% assume it matures at July 10 2028 and that we are at March 5, 2013.



As can be seen, we are 'today' (5/3/2013) somewhere between two dates of interest payments. So if you would sell this bond, then you would want to receive (1) the value of the bond at the last coupon date (10/01/2013) plus (b) the accrued interest between today (5/3/2013) and that most recent coupon date (because that interest was earned while you owned the bond). The clean price or quoted price is the value at the most recent coupon date, the dirty price or cash price is the clean price increased with the accrued interest. Note that for the computation of the accrued interest the daycount convention must be taken into account.

The second particularity is that, as usual with a future, the bond in the future contract is highly standardised, i.e. with a fixed face value, a fixed interest (inclduing compounding) and a fixed maturity. Therefore, the bond specified in the contract does not always exist in the market, so the underlying is "artificial" or "synthetic".

Example 2.1. Assume a synthetic bond that matures in 5 years, has a nominal value of 100 and a (annualy compounded) interest rate of 6%, interests are paid annually.

If todays interest rate is 10%, then the value of this bond today is $\sum_{t=1}^{5} \frac{100\times 0.06}{(1+0.1)^t} + \frac{100(1+0.06)}{(1+0.1)^5} = 88.56$.

The example shows that computing the value of a synthetic bond poses no problems. However, if we want to find the price in the contract, then we (1) borrowed money and bought the asset and (b) shorted a future. The second point (b) is not a probem, but (a) is a problem because one can not buy a synthetic bond (it is artificial).

Therefore, at the creation of a future contract it has associated with it a basket of deliverable bonds that exist in the market, and each of these deliverable bonds has a conversion factor that makes its value equal to the value of the "artificial" bond. The conversion factors are computed based on the clean price of the bonds. The conversion factor is such that the price of the deliverable bond is equal to the price of the synthetic bond (=100 when the contract is created) times the value of the deliverable bond of $100 \times CF_d = P_{d,t=0}$ for each deliverable bond d in the basket , where $t = 0$ is the time the future on the bond is created. [6].

At the settlement time of the future (e.g. at T) the price in the contract is the value of the synthetic bond at T, let's call it SY_T, it is the price of an artificial, non-existing bond. The one with the short position will, at T have to sell the undering at SY_T (per unit, i.e. per 100 nominal), note that $SY_0 = 100$ and SY_T can be computed from the contract and the interest rates at T.

So the one with the short position can buy any deliverable in the basket, and sell it at $CF_d \times SY_T$, of course he has to buy the bond on the spot at $P_{d,t=T}$ for all deliverables in the basket. In other words, his profit at T is $CF_d \times SY_T - P_{d,t=T}$ and this for all deliverable d in the basket of underlying bonds. He will try to maximise his profit so he will choose the underlying deliverable bond that costs less, or he chooses the cheapest-to-deliver bond.

To be more precise, the price of the underlying will be the the most recent settlement price of the future contract times the conversion factor plus the accrued interest.

Note that all this is because of the mismatch between the underlying and the asset to deliver, we wiil see that this will be termed as "basis risk".

So, if we know the cheapest to deliver and the delivery date, then we will at $t = 0$ buy the cheapest to deliver at S_0, we will receive an income I that is the present value on the coupons of the cheapest to deliver and therefore the price in the contract should be $F_0 = (S_0 - I)e^{rT}$.

The difficulty with this formula is that at $t = 0$ the cheapest to deliver is not known, therefore I must be estimated.

2.6.5 Duration based hedging using interest rate futures.

We have seen the definition of duration in section 1.4.3. The duration was the average time you have to wait until you receive cash payments.

Let V_F be the contract price in the futures contract at maturity and D_F the duration of the futures contract, then (see section 1.4.3) $dV_F = -V_F \cdot D_F \cdot dy$.

Let P be the value of the bond portfolio (at maturity) and D_P the duration of it, then (see section 1.4.3) $dP = -P \cdot D_P \cdot dy$.

So when the interest rate changes by dy, then the contract price changes by dV_F and the prodtflio value by dP, so to hedge this change we have to use N future contracts such that N future, N choosen such that $NdV_F = dP$. Subsituting the above results for dV_F, dP we find that $-N \cdot V_F \cdot D_F \cdot dy = P \cdot D_P \cdot dy$ or $N = \frac{P \cdot D_P}{V_F \cdot D_F}$

Note that we want to know it to define the hedge at a certain date, while the values V_F, P are at maturity. V_F will obviously depend on the value that is choosen as "cheapest-to-deliver" which is known at maturity but not at the moment that the hedge is put in place.

[6] Note that the conversion factor is bigger than 1 if the coupon of the deliverable bond is higher than the coupon of the synthetic bond.

2.7 Basis (risk)

2.7.1 Definition of basis.

A forward can be used to hedge the uncertainty of the price of an asset at some date in the future. If you own the asset and you short a forward with exactly the same underlying asset, then the risk was hedged away.

In many cases the asset that one owns is noet exactly the same as the one in the foward contract. Let $S_{t,c}$ be spot price of the asset in the contract and $S_{c,o}$ the spot price of the asset that you own.

At maturity the forward price will be $F_{T,c} = S_{T,c}$ and for $0 \leq t < T$ $F_{t,c} = S_{0,c}e^{r(T-t)}$. If the asset that we own is not exactly the one in the contract, then at T the value $S_{T,o}$ me differ from $S_{T,c}$ and the owning of the asset is not perfectly hedged by the short position in the forwad.

Definition 14. *The basis is the difference between the spot price and the forward price at t, i.e.* $b_t = S_t - F_t$.

Note that of the asset owned and the one in the contract are identical, then $F_T = S_T$ *and the bsis at* T *is zero. If both are not identical then* b_T *can be different from zero.*

For $t < T$ *the basis* b_t *can also be different from zero.*

2.7.2 Decomposition of the basis.

As an example let us assume that we own an asset A_o at t_1 and to hedge the price risk at t_2 we short a future at t_1 with an underlying asset A_c and a maturity t_2.

At t_2 we sell the asset and receive $S_{t_2,o}$ and from the future we have (short position) $X - F_{t_2,c} = F_{t_1,c} - F_{t_2,c}$ or in total $S_{t_2,o} + F_{t_1,c} - F_{t_2,c}$. By defintion of the basis is the difference between the spot price of the asset you own ($S_{t_2,o}$) minus the future price at the same time ($F_{t_2,c}$). So in total we have at t_2 $F_{t_1,c} + b_{t_2}$.

Note that $b_2 = S_{t_2,o} - F_{t_2,c} = S_{t_2,o} \overbrace{-S_{t_2,c} + S_{t_2,c}}^{=0} - F_{t_2,c}$.

In this way we can decompose the basis in two sources of deviation:

$$b_2 = \underbrace{S_{t_2,o} - S_{t_2,c}}_{\substack{\text{difference due to}\\ \text{asset mismatch}}} + \underbrace{S_{t_2,c} - F_{t_2,c}}_{\substack{\text{difference if assets}\\ \text{would be identical}}}$$

Note that the second term is zero when $t_2 = T$.

Definition 15. *Hedging using non-identical assets is called cross-hedging.*

Obviously the price of bith asset should have a high correlation.

An example of cross-hedging is for an airline company; if they want to hedge the risk of their jet fuel price, they should ideally use futures on jet fuel. However, futures on Jet fuel are not traded, theirfore they use futures on heating oil.

2.7.3 Hedging the basis; determining the minimum variance hedge ratio.

If the assets that you hedge and the ratio in the forward are identical, then we have seen that Δ is 1 or that we can eliminate risk by combining a short position in a forward with a same amount of stock. So the ratio of the number of forwards to the number of stock is 1. If the assets are not identical then a hedge ratio of 1 is not always optimal.

Let us assume the the optimal hedge ratio is h. Then we have h units stock for every unit in the forward, and the value of the portfolio is $S_t - hF_t$. The hedge ratio will be choosen such that it minimises the variance. The variance of $S_t - hF_t$ is[7] $V = \sigma_S^2 + h^2\sigma_F^2 - 2h\rho\sigma_S\sigma_F$.

The minimum is found as $\frac{dV}{dh} = 0$ or $2h\sigma_S^2 - 2\rho\sigma_S\sigma_F = 0$ or the optimal value for h, h^* is

$$h^* = \rho\frac{\sigma_S}{\sigma_F}$$

This is the minimum variance hedge ratio.

If we substitute this in the above formula for V then we find that the minimal variance is

$$V^* = \sigma_S^2 + (\rho\frac{\sigma_S}{\sigma_F})^2\sigma_F^2 - 2(\rho\frac{\sigma_S}{\sigma_F})\rho\sigma_S\sigma_F = \sigma_S^2(1-\rho^2)$$

The share of the variance that remains is $\frac{V^*}{\sigma_S^2}$ and the hedge effectiveness ratio or the risk reduction ratio is a measure of the variance that was hedged away. It is defined as

$$1 - \sqrt{\frac{V^*}{\sigma_S^2}} = 1 - \sqrt{1-\rho^2}$$

2.7.4 Optimal number of contracts.

In the section before we assumed that the amount of asset in the contract was 1 and that we had to hedge one unit of stock. If the number of units of asset in the contract is Q_c and the number of units of stock to hedge is Q_o then the number of contracts to hedge the Q_o units will be

$$N^* = h^*\frac{Q_o}{Q_c}$$

because each future contract covers Q_c units and we have Q_o units, so we need $\frac{Q_o}{Q_c}$ contracts to hedge with identical assets. If there are differences then you need to correct with h^* as explained before.

2.7.5 Hedging a floating rate loan

To do

[7] $\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2 + 2\rho\sigma_x\sigma_y$

3 Swaps

3.1 Definition of a swap

Definition 16. *A swap is an over-the-counter (OTC) agreement between two parties to exchange cash flows at specified times in the future, according to specified rules.*

The elements in a swap contract are:

- *The **cash flows**; the values of the cash flows are not themselves in the contract, they have to be derived from the rules (e.g. in an interest rate swaps the cash flows will be computed from interest rates defined in the contract);*
- *the **dates at which the cash flows occur**;*
- *The **rules** for computing cash flows and for discounting.*

Depending on on the rules that define the value of the cash flows one can distinguish between:

- ***Interest rate swaps**; the cash flows result from interest payments;*
- ***Currency swaps**; the cash flows result from exchange rates and or interests on different currencies,*
- *...*

Remark 3.1. *A forward contract is a special case because it is an exchange of cash flows.*

3.2 Types of swaps.

3.2.1 Interest rate swaps (IRS)

Definition 17. *In an interest rate swap, the cash flows result from interest payments where one part is a flow from a fixed rate and the other one (in the opposite direction) from a floating rate.*

Market makers quote the fixed rate.

*The **bid rate** is the fixed rate the market maker pays in exchange for a receiving floating rate;*

*The **offer rate** is the fixed rate the market maker receives in exchange for paying a floating rate;*

*The **swap rate** is the average of the bid and the offer rate.*

3.2.2 Currency swaps

see later

3.3 Valuation of a IRS swap using discounted cash flows

.

3.3.1 Net present value of the cash flows.

Property 12. *The valuation of a swap should not be too difficult because it is just an exchange of cash flows at different dates in the furture. Therefore the value of the swap can be computed when*

1. ***The cash flows are known****; these have to de derived from the rules in the contract;*
2. ***The dates at which the cash flows occur are known****; these are in the contract;*
3. ***The discounting frequency and discounting rate are known****.*

Let's analyse a swap that pays a fixed rate between $[0, T]$ in exchange for a floating rate between $[0, T]$ at dates $t_0 = 0, t_1, t_2, \ldots, t_n = T$.

Moreover, the discount factors (for compounding frequency m) are $DF^m(0, t_i)$.

- after one period, so at t_1 we have the following flows:
  - we pay an amount $fix_1 = Q \times S_{0,T} \times (t_1 - t_0)$, note that the interests are paid after each period, so we do do not coumpound here !;
  - we receive an amount $float_1 = Q \times F_{t_0,t_1} \times (t_1 - t_0)$
- after two periods, so at t_2 we have the following flows:
  - we pay an amount $fix_2 = Q \times S_{0,T} \times (t_2 - t_1)$, note that the interests are paid after each period, so we do do not coumpound here !;
  - we receive an amount $float_2 = Q \times F_{t_1,t_2} \times (t_2 - t_1)$
- ...
- after n periods, so at $t_n = T$ we have the following flows:
  - we pay an amount $fix_n = Q \times S_{0,T} \times (t_n - t_{n-1})$, note that the interests are paid after each period, so we do do not coumpound here !;
  - we receive an amount $float_n = Q \times F_{t_{n-1},t_n} \times (t_n - t_{n-1})$

All these flows occur at different points in time, therefore, if we want to know the value today, we have to discount them and then add them and the value for the **"pay fix receive float"** IRS is:

$$V_{pfi.rfl} = \sum_{i=1}^{n} (F_{t_{i-1},t_i} - S_{0,T}) \times DF^m(0, t_i) \times Q \times \Delta t \qquad (13)$$

The value of a**"pay float receive fix"** IRS is:

$$V_{pfl.rfi} = \sum_{i=1}^{n} (S_{0,T} - F_{t_{i-1},t_i}) \times DF^m(0,t_i) \times Q \times \Delta t \qquad (14)$$

3.3.2 Determining the initial swap rate.

Property 13. *Initially, because of arbitrage arguments, the value of the swap is (as was the case for forwards, forward rate agreements, ...) is* ***equal to zero****.*

Obviously this only holds for the initial value only, at any time after that the IRS will have a value different from zero.

From the above equations we can find that $S_{0,T}$ can be computed from $0 = Q \times \Delta t \times \left(\sum_{i=1}^{n}(F_{t_{i-1},t_i} - S_{0,T}) \times DF^m(0,t_i)\right)$ or $\sum_{i=1}^{n}(F_{t_{i-1},t_i} - S_{0,T}) \times DF^m(0,t_i) = 0$. This implies that $\sum_{i=1}^{n} F_{t_{i-1},t_i} \times DF^m(0,t_i) = \sum_{i=1}^{n} S_{0,T} \times DF^m(0,t_i)$

We find that:

$$S_{0,T} = \frac{\sum_{i=1}^{n} F_{t_{i-1},t_i} \times DF^m(0,t_i)}{\sum_{i=1}^{n} DF^m(0,t_i)}$$

Except for $i = 0$, the F_{t_{i-1},t_i} are unknown at $t = 0$, so we have to "replace" them by the forward rates f_{t_{i-1},t_i} the we can derive from the discount factors and those can be computed at $t = 0$:

$$S^m_{0,T} = \frac{\sum_{i=1}^{n} f^m_{t_{i-1},t_i} \times DF^m(0,t_i)}{\sum_{i=1}^{n} DF^m(0,t_i)} \qquad (15)$$

Property 14. *The swap rate for a maturity T, i.e. the fixed rate $S_{0,T}$ is computed under the assumption that the forward rates are realised. These forward rates can be derived from the discount factors (with a certain compounding frequency).*

These swap rates depend on the compounding frequency m.

Remark 3.2. *Note that there is a swap rate for every maturity T and for any compounding frequency.*

Remark 3.3. *Note that the fixed (and the flowting) swap rates are paid every month, so interest payments using $S_{0,T}$ are not compounded and can NOT be used to compute discount factors !!. This is what distinguishes them from the spot rates $R_{0,T}$.*

Remark 3.4. *The formula 15 writes the swap rates as a function of the discount factors and the forward rates.*

The formula can also be used to derive discount factors from swap rates, but only if we know the swap rates for all the maturities !!

3.3.3 Valuation of an IRS as a difference of two bonds.

Note that, by slightly changing the reasoning in section 3.3.1 we can also derive the initial value of an IRS in another way. Indeed, let us look wat happens when we pay, on top of interest flows, also the notional amount Q at the end. Then the result is the same, except that in $t_n = T$ there are two additional flows:

- We pay the amount Q with the fixed rate in T
- We receive the amount Q with the floating rate in T

But, after that (see supra) we take the difference and as these two amounts are identical we will of course find the same value, so the value is also equal to:

$$V_{pfi.rfl} = \sum_{i=1}^{n}(F_{t_{i-1},t_i} - S_{0,T}) \times DF^m(0,t_i) \times Q \times \Delta t + Q \times DF(0,T) - Q \times DF(0,T)$$

But then we can see that this is a difference of two terms:

- $Q \times DF(0,T) + \sum_{i=1}^{n} F_{t_{i-1},t_i} \times DF^m(0,t_i) \times Q \times \Delta t$
- $Q \times DF(0,T) + \sum_{i=1}^{n} S_{0,T} \times DF^m(0,t_i) \times Q \times \Delta t$

But if you analyse this then the first term is just the sum of discounted interest payments and the dicounted principal amount on a bond with floating interest rate, while the second one is the same for a fixed rate. Note that in both cases the interests are paid monthly and NOT compounded !

Property 15. *So we find that the value of an IRS that pays a fixed rate in exchange for a floating rate can also be written in terms of bond prices (with periodic payments of interests, so not compounded):*

$$V_{pfi.rfl} = B^{float} - B^{fix}, \text{ interests paid every period} \tag{16}$$

And the swap rate is the value of the fixed rate for which $B^{fix} = B^{float}$

Let us look at the floating bond in more detail; at t_i you receive an interest per euro that is equal to (taking the compounding frequency into account) $(1 + \frac{f_{t_{i-1},t_i}}{m})^{m\Delta t} - 1$ (minus one because you only get the interest !). Using the schema in figure 1 we know that $(1 + \frac{R_{0,t_{i-1}}}{m})^{m(t_{i-1}-0)}(1 + \frac{f_{t_{i-1},t_i}}{m})^{m(t_{i-1}-t_i)} = (1 + \frac{R_{0,t_i}}{m})^{m(t_i-0)}$ of in terms of discount factors $\frac{1}{DF(0,t_{i-1})}(1 + \frac{f_{t_{i-1},t_i}}{m})^{m(t_{i-1}-t_i)} = \frac{1}{DF(0,t_i)}$ so

$$\left(1 + \frac{f_{t_{i-1},t_i}}{m}\right)^{m(t_{i-1}-t_i)} = \frac{DF(0,t_{i-1})}{DF(0,t_i)}$$

To get the interest on one euro we have to subtract 1 so at t_i we get an interest of $\frac{DF(0,t_{i-1})}{DF(0,t_i)} - 1$ euro at t_i, so if we want to have the value of this at $t = 0$ we have to discount so to multiply by $DF(0,t_i)$.

So the discounted value of the amount interest received on 1 EUR of the bond is

$$\left(\frac{DF(0,t_{i-1})}{DF(0,t_i)} - 1\right) DF(0,t_i) = DF(0,t_{i-1}) - DF(0,t_i) \tag{17}$$

So the present value of the floating bond that pays interests every period is

$$\begin{aligned} &\overbrace{DF(0,0) - DF(0,1)}^{period1} + \overbrace{DF(0,1) - DF(0,2)}^{period2} + \\ &\cdots + \overbrace{DF(0,T-1) - DF(0,T)}^{periodT} + \overbrace{DF(0,T)}^{principal} \\ &= DF(0,0) = 1 \end{aligned}$$

Property 16. *The swap rate can also be determined as the value that makes the value of a boond paying fixed periodic interest equal to its principal amount, of as a solution of* $B^{fix} = Q$.

3.3.4 Computing discount factors from swap rates & vice versa.

Assume that we know the swap rates $S^m_{0,t}$ for different maturities t. The goal is to derive the discount factors from $S_{0,t}$. Note that the swap rates $S_{0,T}$ can not be used directly to find the discount factors because there is no compounding in $S_{0,T}$!!

How can we proceed ?

Let us take a swap with maturity t and a given swap rate $S^m_{0,t}$. At time $t = 0$ the swap has a value of zero and this is how we found $S^m_{0,t}$. By the above property, we know that $S^m_{0,t}$ is the value the makes the periodic interest payments plus the principal, all discounted, equal to the face value of a fixed bind, so:

- $t = t_1$: $F = F \times S^m_{0,t_1} \times (t_1 - 0) \times DF^m(0,t_1) + F \times DF^m(0,t_1)$

  so we find that

  $DF^m(0,t_1) = \frac{1}{1+S^m_{0,t_1}(t_1-0)}$

  $S^m(0,t_1) = \frac{1-DF^m(0,t_1)}{1+DF^m(0,t_1)(t_1-0)}$

- $t = t_2$: $F = F \times S^m_{0,t_2} \times (t_1 - 0) \times DF^m(0,t_1) + F \times S^m_{0,t_2} \times (t_2 - t_1) \times DF^m(0,t_2) + F \times DF^m(0,t_2)$

  so we find that

  $DF^m(0,t_2) = \frac{1-S^m_{0,t_2}\times(t_1-0)\times DF^m(0,t_1)}{1+S^m_{0,t_2}(t_2-t_1)}$

  $S(0,t_2) = \frac{1-DF(0,t_2)}{DF(0,t_1)\times(t_1-0)+DF(0,t_2)\times(t_2-t_1)}$

- ...

3.4 Very important overview on interest rate swaps.

3.4.1 Overview on important concepts.

First of all it is noted that a swap is a series of cash flows. An interest rate swap is a special case (but an important one !).

Very special in an interest rate swap is that the interests are paid out periodically, so when we talk about an interest rate swap we have **no compounding** ! The difference between an interest rate swap and z zero coupon bond is illustrated in figure 2, the zero coupon bond is in the upper panel, the interest rate swap in the middle panel.

> Only the $\Gamma^m_{0,t}$ can be used to compute discount factors !

The value of the swap can be computed as the sum of the discounted discounted cash flows or as a difference of the values of two bonds.

To find these values you need cash flows $\Delta I_{t_{i-1},t_i}$ which is the difference between an interest (in EUR) of a fixed rate and an interest (in EUR) of a floating rate and the discount factors to discount these values to today $t = 0$. So the value of the swap that pays a fixed rate and receives a floating rate is

> $V = \sum_i (I^{float}_{t_{i-1},t_i} - I^{fix}_{t_{i-1},t_i}) \times DF^m(0, t_i)$

The fixed rate is known in the contract, just as the face value, so we can compute I^{fix}. To compute the I^{float} we have to know the floating rates for $t > 0$, as this is in the future we do not know them.

> We will use the forward rates f_{t_1,t_2} to compute the I^{float}.

The value of the swap can also be derived as the difference of the values of a bond that pays a floating rate mines one that pays a fixed rate interest. These

> Such bonds are not a zero coupon bond, as illustrated in figure 2

> So V can also be computed as $V = V^{float} - V^{fix}$.

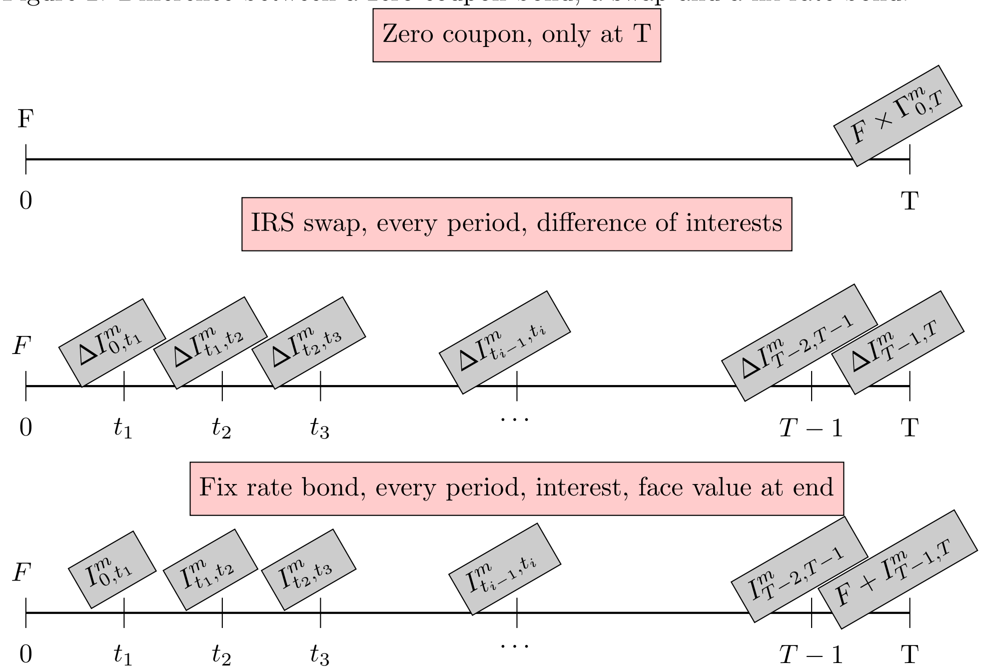
For computing the forward rates and the for discounting the interest cash flows we will need the discount factors.

> You can NOT use swap rates directly to compute discount factors.

The swap rate is the fixed rate that makes the intial value of the swap equal to zero. From the above it follows that, to find the swap rate, you need the value of the bond or of the discounted cash flows of the swap. Conequently,

> The swap rate is the value of the fixed rate such that $V = 0$. It depends indirectly on the discounting frequency in two ways;
>
> 1. via the discount factors and
> 2. via the computation of the interest between $[t_{i-1}, t_i]$
>
> ASK THE PROFESSOR WHETHER THE SECOND 'VIA' MAKES SENSE, BECAUSE IT HAS AN IMPACT ON THE COMPLEXITY AND ON THE SOLUTION OF EXERCISE Q3.2 !

Figure 2: Difference between a zero coupon bond, a swap and a fix rate bond.



3.4.2 Discount factors and spot price for compounded interest.

$$\Gamma^m(t_0, t) = \left(1 + \frac{R^m_{t_0,t}}{m}\right)^{m(t-t_0)} \tag{18}$$

$$\Gamma^c(t_0, t) = e^{R^c_{t_0,t}(t-t_0)} \tag{19}$$

$$\Gamma^s(t_0, t) = \big((1 + R^s_{t_0,t}(t - t_0)\big) \tag{20}$$

for $t_0 = 0$ these are (compounded) interest rates, for $t_0 > 0$ these are (compounded) forward rates.

Property 17. *To switch between disocunt rates and interest rates use:*

$$DF^*(t_0, t) = \frac{1}{\Gamma^*(t_0, t)} \implies R^*_{t_0,t} \longleftrightarrow DF^*(t_0, t) \tag{21}$$

Property 18. *The value of the interest (in EUR) between two periods t_1, t_2 for a face value F is equal to:*

$$I^*_{t_1,t_2} = \Gamma^*_{t_1,t_2} \times F - F = (\Gamma^*_{t_1,t_2} - 1) \times F \tag{22}$$

Note that this depends on the compounding frequency !

3.4.3 Forward rates and discount factors.

Using the schema in figure 1 we can see that $\frac{1}{DF^*(0,t)} \frac{1}{DF^*(t,T)} = \frac{1}{DF^*(0,T)}$

Property 19. *Forward rates can be computed using :*

$$\frac{1}{DF^*(0,t)} \frac{1}{DF^*(t,T)} = \frac{1}{DF^*(0,T)} \implies f^*_{t,T} \tag{23}$$

3.4.4 Swap rates and discount factors.

You can NOT use swap rates for discounting, however, if the swap rates for different maturities are known, then one can compute discount rates from swap rates and vice versa.

Assume that today ($t = 0$) we know the swap rates for multiple maturities, i.e. we know S^*_{0,t_i} for several t_i. The '*' superscript indicates the indirect dependence of the swap rates on the compounding frequency.

Let there cash flows of the swap appear at $t_1 < t_2 < t_3 < \cdots < T$.

- We know S^*_{0,t_1} and we know also that the swap rate is the value of the fixed rate that makes the present value of the interest payments of a fixed rate bond (with periodic payments, as in figure 2) equal to the face value.

  So we know that $F = F \times (\Gamma^*_{0,t_1} - 1) \times DF^*(0,t_1) + F \times DF^*(0,t_1)$ or $1 = (\Gamma^*_{0,t_1} - 1) \times DF^*(0,t_1) + DF^*(0,t_1)$ or

  $$DF(0,t_1)^* = \frac{1}{\Gamma^*_{0,t_1}}$$

  So in case of

  - continuous compounding we have $\Gamma^*_{0,t_1} = e^{S^c_{0,t_1}(t_1-0)}$ (because the interests between $[0,t_1]$ are computed with S_{0,t_1} and then paid !) so we find: $DF^c(0,t_1) = e^{-S^c_{0,t_1}(t_1-0)}$
  - discrete compounding we have: $DF^m(0,t_1) = (1 + S^m_{0,t_1}/m)^{-m(t_1-0)}$
  - simple compounding we have $DF^s(0,t_1) = (1 + S^s_{0,t_1} \times (t_1 - 0))^{-1}$

These formulas can be used to find the discount factor $DF^*(0,t_1)$ when S^*_{0,t_1} is known and vice versa.

- If we also, besides S^*_{0,t_1}, know S^*_{0,t_2} we reason in a similar way:

  $$F = \overbrace{\overbrace{F \times (\Gamma^*_{0,t_2} - 1)}^{\text{interest after t1}} \times DF^*(0,t_1)}^{\text{discounted from t1 to 0}} + \overbrace{\overbrace{F \times (\Gamma^*_{0,t_2} - 1)}^{\text{interest after t1}} \times DF^*(0,t_2)}^{\text{discounted from t2 to 0}} + \overbrace{F \times DF^*(0,t_2)}^{\text{disc. face value from t2 to 0}}$$

  or

  $1 = (\Gamma^*_{0,t_1} - 1) \times DF^*(0,t_1) + (\Gamma^*_{0,t_2} - 1) \times DF^*(0,t_2) + DF^*(0,t_2) = (\Gamma^*_{0,t_1} - 1) \times DF^*(0,t_1) + \Gamma^*_{0,t_2} \times DF^*(0,t_2)$ or

$$DF(0,t_2)^* = \frac{1-(\Gamma^*_{0,t_1}-1)\times DF^*(0,t_1)}{\Gamma^*_{0,t_2}}$$

These formulas can be used to find the discount factor $DF^*(0,t_1)$ when S^*_{0,t_1} and S^*_{0,t_2} is known, because $DF^*(0,t_1)$ is known from the previous step and vice versa, we can compute $S_{0,t}$ when the discount factors are known.
DO NOT LEARN THESE FORMULAS BY HEART, JUST REMEMBER THE LOGIC, THE EXERCISS WILL SHOW THAT IT IS EASY !

- ...
- At $t = t_k$, simplifying to "simple compounding":

$$F = \overbrace{F \times S^s_{0,t_k} \times (t_1 - 0) \times DF^s(0,t_1)}^{\text{discounted interest t1}} \tag{24}$$

$$+ \overbrace{F \times S^s_{0,t_k} \times (t_2 - t_1) \times DF^s(0,t_2)}^{\text{discounted interest t2}} \tag{25}$$

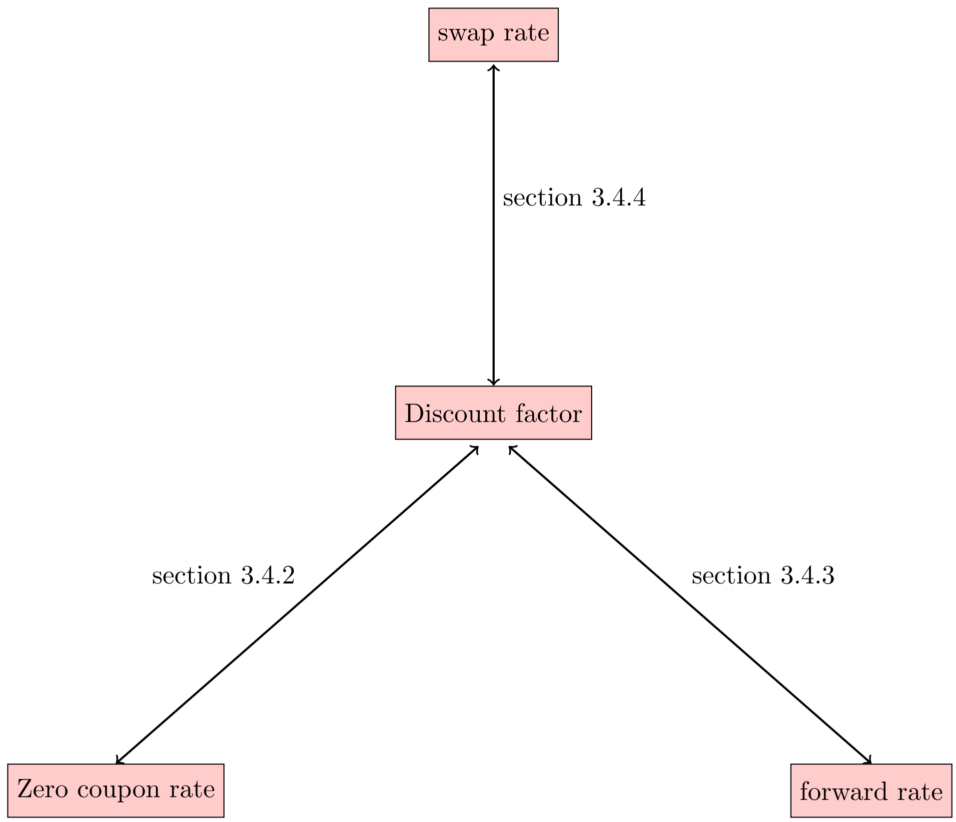
$$+ \ldots \overbrace{F \times S^s_{0,t_k} \times (t_k - t_{k-1}) \times DF^s(0,t_k)}^{\text{discounted interest tk}} \tag{26}$$

$$+ \overbrace{F \times DF^s(0,t_k)}^{\text{discounted principal tk}} \tag{27}$$

but the formula taking into account all kinds of compounding is obtained when $S_{0,t}$ is replaced by $\Gamma^*_{0,t} - 1$.

This procedure were we compute the values one after the other is called bootstrapping.

3.4.5 NEW: Schematic overview.



3.5 Currency swaps (foreign exchange swaps)

3.5.1 What is a currency swap ?

Currency swaps are very similar to interest rate swaps but there are fundamental differences.

An interest rate swap has (simplified) a principal amount in one currency and (if it is a fix-for-float interest swap) a fixed rate and a floating rate. The differences in cash-flows are exchanged periodically, the principal amount is not exchanged, not at the beginning nor at the end.

A currency swap has the initial prinicipal expressed in two currencies (e.g. once in dollar and once in euro) and for each of the two currencies an interest rate is fixed. Therefore one can periodically compute the interest on each currency, so we get a series of cash flows for each currency or "for each leg of the swap". What is different from an IRS is that the principal amounts are exchanged ! just as well at the beginning as at maturity.

Assume that two parties agree in a currency rate swap: party 1 will receive USD (which implies that at the start it has to buy USD from party 2) and pay EUR and party 2 will receive EUR (and therefore at the beginning it will buy EUR from party 1) and pay USD.

If you buy another currency then you need an exchange rate, at $t = 0$ let $E_0^{EUR/USD}$ be the number of euros for one USD, and $E_0^{USD/EUR}$ the number of USD per EUR. Then it holds that[8] $E_0^{EUR/USD} = \frac{1}{E_0^{USD/EUR}}$. E stands for exchnage rate, as a superscipt we indicate in which direction we exchange.

We also need the quantities, let's assume it is about Q_0^{EUR} euros (e.g. Q_0^{EUR}=1 million eur), then obviously, as the principal amounts are exchanged initially the amount in USD will be $Q_0^{USD} = Q_0^{EUR} \times E_0^{USD/EUR}$ where $E_0^{USD/EUR}$ is the exchange rate of the USD per EUR. To be more precise, we will use the exchange rate that is known at the time the swap is initiated, i.e. $E_0^{USD/EUR}$.

So now we have a swap contract with:

- An amount expressed in two currencies: Q_0^{EUR} and Q_0^{USD} where $Q_0^{USD} = Q_0^{EUR} \times E_0^{USD/EUR}$.

  These two amounts have (at $t = 0$) the same value, else one of the two parties would not be willing to conclude the deal !

  It is important to note that these amounts are in the contract, so their conversion in another currency is based on the exchange rate at $t = 0$

- an interest rate on each of the two currencies, let's assume it is a fix-for-fix currency swap (but it is easy to generalise to other types). Then we have an interest rate for each currency so r_{USD} and r_{EUR}. note that these are not necessarily the risk free rates, so we will later on have a discount rate as well.

- A periodicity for exchanging the cash flows, assume that after each Δt the flows are swapped (=exchanged).

- the interests are exchanged at T,

- the principals are exchanged at T,

  It is important to note that these cash flows are computed on the amounts that are in the contract so we use Q_0^{USD}, Q_0^{EUR}.

3.5.2 Cash flows in a currency swap.

Let us schematise this in the table below (the explanation is after the table)

The first column represents the times at which cash flows occur, below party 1 we have both legs of the cash flows for party 1 and similar for party 2. The left leg is the USD leg, the right one the EUR-leg (and vice versa for party 2).

Note that the first line, $t = 0$ is there only "as information". For each party the value of the exchanged principals is zero by the way we constructed it using the interest rate !

[8] If 1 USD costs 2 EUR then 1 EUR will cost 1/2 USD.

time	party P_1		party P_2	
$t = 0$	Q_0^{USD}	$-Q_0^{EUR}$	$-Q_0^{EUR}$	$+Q_0^{USD}$
$t = \Delta t$	$+Q_0^{USD} \times r_{USD}$	$-Q_0^{EUR} \times r_{EUR}$	$+Q_0^{EUR} \times r_{EUR}$	$-Q_0^{USD} \times r_{USD}$
$t = 2\Delta t$	$+Q_0^{USD} \times r_{USD}$	$-Q_0^{EUR} \times r_{EUR}$	$+Q_0^{EUR} \times r_{EUR}$	$-Q_0^{USD} \times r_{USD}$
$\vdots$	$\vdots$	$\vdots$		
$t = T$	$+Q_0^{USD} \times r_{USD}$ $-Q_0^{USD}$	$-Q_0^{EUR} \times r_{EUR}$ Q_0^{EUR}	$+Q_0^{EUR} \times r_{EUR}$ $-Q_0^{EUR}$	$-Q_0^{USD} \times r_{USD}$ Q_0^{USD}

1. At $t = 0$ party 1 buys Q_0^{USD} in exchange for Q_0^{EUR} euro from party 2 and vice versa. Obviously the two parties will not agree if these amounts, after conversion to a common currency, would be different.

   So $Q_0^{EUR} = Q_0^{USD} E_0^{EUR/USD}$,

   or, because$E_0^{USD/EUR} = \frac{1}{E_0^{EUR/USD}}$, $Q_0^{USD} = Q_0^{EUR} E_0^{USD/EUR}$

   Note that at $t = 0$ the exchange rate is known, but all the later exchange rates are not known at $t = 0$!

   We have to be very explicit because the exchange rates change over time; the amounts in the contract are fixed at $t = 0$ using the exchange rates valid at $t = 0$, therefore we explicitly mention the subscript '0'.

   Note: for both parties the sum is zero at $t = 0$!

2. At $t = \Delta t$ we have the first cash flow:

   - party 1 receives interest on the USD that it has received at the start, i.e. it receives $+Q_0^{USD} \times r_{USD}$.
   - On the other hand, party 1 will have to pay interest on the euros, so $-Q_0^{EUR} \times r_{EUR}$.

   So party 1 receive a cash flow $Q_0^{USD} \times r_{USD} - Q_0^{EUR} \times r_{EUR}$. As these amounts are in different currencies we can not make the substraction !. Therefore we keep the amounts for each leg separate !

   For party 2 the opposite will happen.

3. At $t = 2 \cdot \Delta t$ we have the second cash flow, in a completely similar way we find the values in each leg.

4. ...

5. At $t = T$ we have a cash flow with two components:

   - The cash flow resulting from the last interest payment at $t = T$:
   - The exchange of the principal amounts Q_0^{USD} and Q_0^{EUR}; these are exchanged in the opposite direction as at $t = 0$.

Very important:

At e.g. $t = \Delta t$ we have two cash flows, one for each leg: So party 1 receives a cash flow $Q_0^{USD} \times r_{USD} - Q_0^{EUR} \times r_{EUR}$. As these amounts are in different currencies we can not make the substraction !.

In order to convert to a common currency we need the exchange rate, however, at $t = 0$ we can impossibly know the exchange rate at $t = \Delta t$ because that is in the future and therefore unknown !

Therefore we convert all the cash flows using the exchange rate at $t = 0$!!!

Note that all cash flows must be discounted to $t = 0$, so we need some estimate of the exchange rates at $\Delta t, 2\Delta t, \ldots$ to compute this at $t = 0$.

Read the example of Hull, section 7.9 if you want an example with concrete numbers !

3.5.3 Valuation of currency swaps.

Note that currency swaps look alike interest rate swaps in the sense that you have two interest rates, the only additional complication is the exchange rate between the two currencies.

Just as for an index rate swap the value of a currency swap can be computed in two ways. One way is to compute the cash flows and discount them, another way is as the difference of two bond values. In stead of EUR and USD we will talk about a domestic currency (D, e.g. D is EUR) and a foreign currency (F, e.g. F is USD).

Let us look at the example in the table:

- We already mentioned that the sum of the two flows is zzero at $t = 0$
- at $t = \Delta t$: we assume that party 1 is in the uero zone (or the EUR is its domestic currency). So both legs for party 1 will be conversted to its domestic currency (EUR in our example).
  - $+Q_0^{USD} \times r_{USD}$ this must be converted to EUR using an exchange rate at $t = \Delta t$. But at $t = 0$ we can not know that exchange rate, so we use $E_0^{EUR/USD}$:

    So for the dollar leg we have at $t = \Delta t$:

    $$+ Q_0^{USD} \times r_{USD} \times E_0^{EUR/USD} \text{ (amount in EUR)}$$

  - For the euro leg no conversion is needed, it is already in euro:

    So for the euro leg we have at $t = \Delta t$:

    $$- Q_0^{EUR} \times r_{EUR}$$

    These amounts should be discounted at a rate r, we assume continuous discounting:

Note that r is the risk free rate in the oney you are working in, we assumed party 1 is working in euro, so r is the risk free rate on euro, which can be different from r_{EUR} that is in the swap contract.
So be very carefull and pay attention to which currency you are working in !

For the dollar leg we have at $t = \Delta t$:

$$(+Q_0^{USD} \times r_{USD} \times E_0^{EUR/USD})e^{-r\Delta t}$$

For the euro leg we have at $t = \Delta t$:

$$-Q_0^{EUR} \times r_{EUR}e^{-r\Delta t}$$

- at $t = 2 \cdot \Delta t$ we find similarly:

  For the dollar leg we have at $t = \Delta t$:

  $$(+Q_0^{USD} \times r_{USD} \times E_0^{EUR/USD})e^{-r\cdot 2\cdot\Delta t}$$

  For the euro leg we have at $t = \Delta t$:

  $$-Q_0^{EUR} \times r_{EUR}e^{-r\cdot 2\cdot\Delta t}$$

- ....
- at $t = T$ we have
  - for the interests:
    For the dollar leg:

    $$(+Q_0^{USD} \times r_{USD} \times E_0^{EUR/USD})e^{-rT}$$

    For the euro leg:

    $$-Q_0^{EUR} \times r_{EUR}e^{-rT}$$

  - for the principal amounts:
    We note that Q_0^{EUR} and Q_0^{USD} have the same value because they were choosen like that in the contract, so we may replace both, and then from the table it follows that
    For the dollar leg:

    $$(+Q_0^{USD} \times E_0^{EUR/USD})e^{-rT}$$

    For the euro leg:

    $$-Q_0^{EUR}e^{-rT}$$

Let us now add up all the discounted cash flows from the USD-leg, note that the common factor $E_0^{EUR/USD}$ can be separated and put in front:

$$\underbrace{E_0^{EUR/USD}}_{\text{exchange rate}}(\underbrace{\overbrace{Q_0^{USD}\times r_{USD}e^{-r\Delta t}}^{\text{discounted interest}}+\overbrace{Q_0^{USD}\times r_{USD}e^{-2r\Delta t}}^{\text{discounted interest}}+\cdots+\overbrace{Q_0^{USD}\times r_{USD}e^{-rT}}^{\text{discounted interest}}+\overbrace{Q_0^{USD}e^{-rT}}^{\text{discounted principal}}}_{\text{value of a abond}})$$

Property 20. *So we find that, if B_{USD} is the (dollar) value of a USD bond with a principal and an interest rate identical to the one in the swap contract, then the value of dollar leg <u>in EUR</u> is equal to $E_0^{EUR/USD}B_{USD}$*

Similarly, if we add up all the discounted cash flows in the euro-leg then we have

$$\underbrace{\overbrace{-Q_0^{EUR}\times r_{EUR}e^{-r\Delta t}}^{\text{discounted interest}}+\overbrace{-Q_0^{EUR}\times r_{EUR}e^{-2r\Delta t}}^{\text{discounted interest}}+\cdots+\overbrace{-Q_0^{EUR}\times r_{EUR}e^{-rT}}^{\text{discounted interest}}+\overbrace{-Q_0^{EUR}e^{-rT}}^{\text{discounted principal}}}_{\text{value of a abond}}$$

Property 21. *So we find that, if B_{EUR} is the (euro) value of a EUR bond with a principal and an interest rate identical to the one in the swap contract, then the value of euro leg <u>in EUR</u> is equal to $-B_{EUR}$*

Property 22. *The value of the swap where dollars are received and euros are paid (see the table, party 1 receives USD) value expressed in EUR is the combination of the two legs so*

$$V_{swap}^{EUR}=E_0^{EUR/USD}B_{USD}-B_{EUR}$$

For party 2 we have the opposite value of course so: The value of the swap where euros are received and dollars are paid then the value expressed in EUR is the combination of the two legs so

$$V_{swap}^{EUR}=B_{EUR}-E_0^{EUR/USD}B_{USD}$$

Note: if at a leter time t you know the exchange rate then you use the most recent, known value for E_t of course.

This is the same result as in Hull on page 169 but everywhere Hull says "dollar" you put "EUR" and what Hull calls "foreign" is "USD" in our example.

3.6 Other topics

3.6.1 Forward rate agreement (FRA).

We already defined a forward rate, $f_{t,T}$ as the interest rate between t and T, where t is in the future. So the forward rate between t_1 and t_2 both in the future is f_{t_1,t_2}.

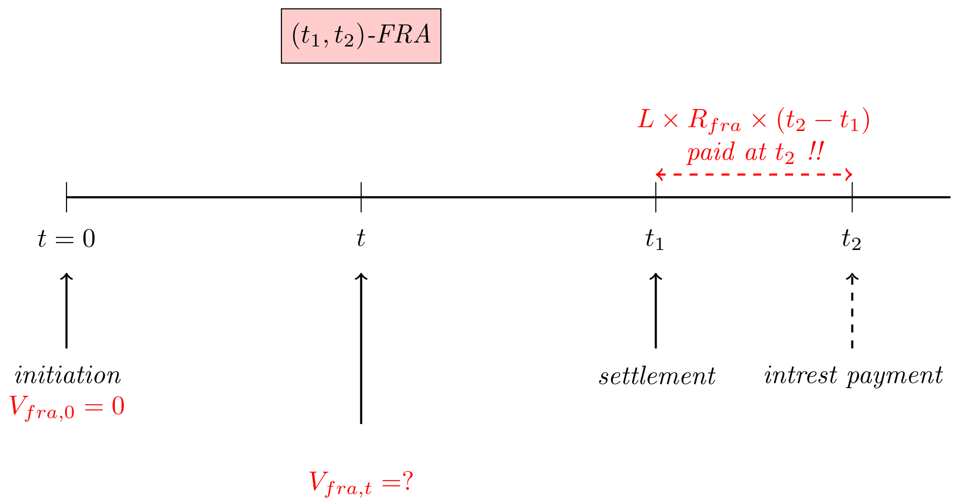
Definition 18. *A forward rate agreement is a forward contract on a loan, i.e. we agree today $t = 0$ to pay interest between t_1 and t_2 on an amount L and the interest rate R_{FRA} and the amount L are fixed in the contract when it is created, just as well as t_1 and t_2.*

In a forward contract you have two positions, a long (when you are the buyer) and a short (you are the seller) one. For an FRA you have a long position if you are the borrower in the loan underlying the forward contract (long is usually the buyer and borrower has the same first letter) in the underlying loan and short if you are the lender in the loan underlying the forward contract.

Obviously, when the FRA is initiated at $t = 0$, the rate in the contract will be choosen at $R_{fra} = f^0_{t_1,t_2}$ (the superscript 0 indicates that we compute the forward rate at $t = 0$.

To make the name complete, a "(t_1, t_2) FRA" is a forward contract with an underlying loan over the period $[t_1, t_2]$ in the future. Note that the length of the loan underlying an "(t_1, t_2) FRA is $t_2 - t_1$!, the interval of the loan is $t_2 - t_1$ and starts at t_1

Note: for an FRA the rates are not continuous and the principal amounts are not exchanged



Remark 3.5. *If I am long in an (3m,9m)-FRA let's say of 5%, then I have the obligation to borrow three months from now, until nine months from now, at 5%.*

If in three months the interest rate is higher e.g. 7%, then, according to the contract I can borrow at 5% , compared to 7% in the market, so someone needing a loan in 3 months and expecting that the interest rates will go up, will have a profit from signing such a long position in a (3m,9m) FRA.

so it is like any other assets, if you are long, you want rates to increase (for any other assets, if you are long, you want prices to increase)

It is important to note that the FRA is concluded at $t = 0$ and that it is a foward with underlying asset being a loan from t_1 to t_2. It is settled at t_1.

We want to find the value of the (t_1, t_2)-FRA for any value of t between $t = 0$ and $t = t_1$ (see figure). We will compute it for the one with a long position, i.e. the borrower in the underlying loan. For the short position you just put a minus sign in front.

The contract is settled at $t = t_1$ so let us assume that the market interest rate is r_1 at t_1. Note that this rate is unknown at t ! (see figure, at t we do not nothing about t_1 because it is in the future !)

According to the contract I can borrow (long position) at a rate R_{fra} , if the market rate is higher, then I have a profit because I can borrow at R_{fra} compared to a higher rate r_1 in the market, the percentage profit is $r_1 - r_{fra}$. Note that this is positive so it is a profit if r_1 is higher, so the sign is fine for the lng position. Note that we do not compound interest in FRAs!

To find the interest that must be paid at t_2 (!!) we have to multiply this by the amount and by $(t_2 - t_1)$ so the interest paid at t_2 is $L(r_1 - R_{fra})(t_2 - t_1)$. This is the value of the FRA, for a long position, at t_2, so to find it at r we have to discount it.

Moreover, we do not know r_1 at t because it is the market rate at t_1 and we do not know it at t. Therefore we replace it with our "best estimate" which is the forward rate computed at t for a loan between t_1, t_2:

Property 23.

$$V^{Long}_{fra,t} = \underbrace{L}_{EUR} \underbrace{(f^t_{t_1,t_2} - R_{fra})}_{a\ percentage} \overbrace{(t_2 - t_1)}^{time\ of\ the\ loan\ in\ year} \underbrace{e^{-r(t_2-t)}}_{discount\ t_2 \to t}$$

In the formula, $DF(t_2, t) = e^{-r(t_2-t)}$ is for continuous compounding, for other compounding frequencies the formula is slightly different !

You do not have to learn this by heart. Just remember the steps that you see in the formula:

1. Determine whether you have a long or a short position;
2. Determine whether you have a profit or loss; if you are long and interests have risen then you have a profit, ... (long positions profit from price increases, see the graphs under forwards: $S_T - K$)
3. compute the difference in interets, if you don't know the market rate then use the forward rate, no compounding is used !
4. compute the length of the loan
5. multiply by the amount

6. discount from $t_2 \to t$.

Remark 3.6. *Note that if we choose R_{fra} equal to the forward rate at $t = 0$ then this formula shows that at the creation of the FRA it holds that $V^{Long}_{fra,0} = 0$. Compare to a forward (future) where the price in the contract is choosen such that at the creation of the contract the value of the contract is zero. This is not a surprise, because an FRA is a special case of a forward contract.*

Remark 3.7. *FRAs are settled at $t = t_1$. Let us see what the value is at settlement; at $t = t_1$ we know the rate R_{t_1,t_2} (it is then no longer a forward rate because we are at t_1, it is however paid at t_2). So at settlement the value of the FRA is $L(R_{t_1,t_2} - R_{fra})(t_2 - t_1)e^{-r(t_2 - t_1)}$*

Property 24. *If one looks at the formula for the value of an FRA we see that it is a face value L times a difference in interest paid $(f^t_{t_1,t_2} - R_{fra})$ times a discount factor. If you compare this two the two cash flow legs in an interest rate swaps, and taking into account that we subtract these two flows, we find that the value of an interest rate swap can also be computed as a series of FRAs !*

Said in another way: an FRA swaps (in one cash flow) a fixed rate (R_{fra}) for a floating rate (the one that you used to compute forward rates, this is usually LIBOR or EURIBOR.)

Note that, in the above example, we borrow L at t_1 and pay $L(1+R_{fra}(t_2-t_1))$ at t_2. We want to find a replicating portfolio. In order to have L at t_1 I have

- to sell (short) today a zero coupon bond with face value L and maturity t_1, its value today is $L \times DF(0, t_1)$,

  When I do this then have L at t_1 (and this L grows to $L \times (1+R_{fra}(t_2-t_1))$ at t_2)

- to buy (long) a zero coupon bond with face value L and maturity t_2 which has value today equal to $-L \times DF(0, t_2)$ (minus because I buy).

  When I do this then I have at t_2 the $L \times (1 + R_{fra}(t_2 - t_1))$ (from the short zero bond) and $-L$ (from the long zer bond)

So at t_2 we have $L \times (1+R_{fra}(t_2-t_1))-L = R_{fra}(t_2-t_1)L$ which is identical to the cash flow of the FRA. As the cash flow is the same, the value of the FRA today must also be equal and we find that it is $L\,(DF(0, t_1) - DF(0, t_2))$

so we find that a (t_1, t_2)-FRA can be hedged by a portfolio of the two above zero coupon bonds !

Note that $DF(0, t_1) - DF(0, t_2) = DF(0, t_2)\left(\frac{DF(0,t_1)}{DF(0,t_2)} - 1\right)$. If we look at the definition of the discount factor $DF(0, t_2)$ with continuous compounding then we find e^{-rt_2} and if we look at how we computed forward rates in property 5 (with simple compounding) we find that $\frac{DF(0,t_1)}{DF(0,t_2)} - 1$ times $t_2 - t_1$ is the other factor in the formula for $V_{fra,0}$ supra.

3.6.2 Overnight index swap (OIS)

3.7 NEW: Forward start swap.

See back of the slides of the professor and exam level question 9 !

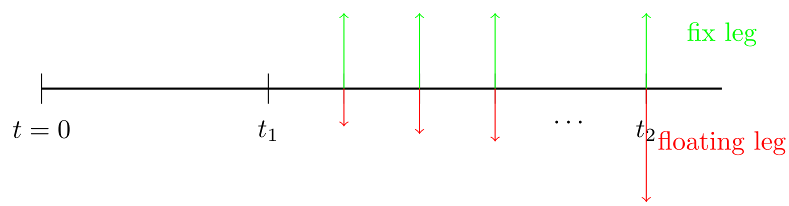
As an example, you are a government and you know that you will need money (liquidity) within 2 years and for a period of 10 years. As todays interest rates are rather favorable and expected to be higher in the future, it is interesting to get an agreement for the future that is based on todays interest rates.

In that case you need a swap that starts at a future date, i.e. a forward start swap or a deferred swap.

Definition 19. *A forward start (or deferred) interest rate swap is just an interest rate swap that starts exchanging cash flows not today but at a future date.*

It can be used when you need a loan in the future and todays interest rates are more favorable. Of course there is a price to pay for this advantage.

The swap starts at t_1 and matures at t_2, so let us call this swap rate S_{t_1,t_2}.



Note that the flows start after t_1 and not at t_1 because interest is paid after passing of Δt.

Obviously, this is a series of cash flows, but not a usual swap because we miss cash flows between $[0, t_1]$!

But we can apply a similar reasoning as for the usual swap to compute the value or the swap rate. For the usual swap we had three methods to compute the value:

1. Compute the present value at $t = 0$ of all the cash flows in the fixed leg, compute the present value at $t = 0$ of the cash flows in the floating leg, and subtract the two.

   The swap rate at $t = 0$ can be computed by equating this difference to zero and to solve the equation for the fixed rate.

2. As the sum of the values of a series of FRAs. If you look at the formula for the value of an FRA, then you can see that this sum is exactlt the same of for the difference of the floating and the fixed legs, but terms are re-ordered and regrouped.

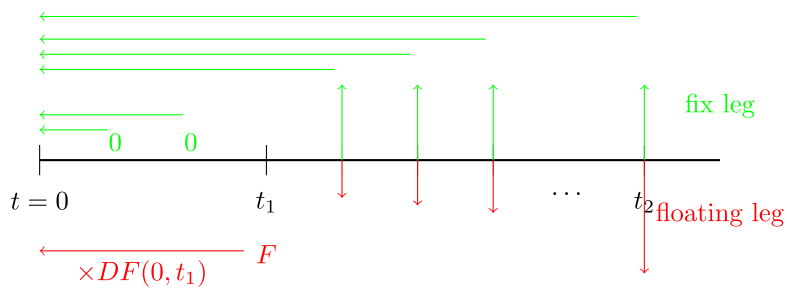
   The swap rate at $t = 0$ can be computed by equating this sum of FRA values to zero and to solve the equation for the fixed rate.

3. As the difference of the values of two bonds. This method will makes computation of the swap rate much faster !

   However, here we have to be "tricky" because we do not have to bonds at $t = 0$ since the interest payments at the beginning are missing. BUT there is a way out and it is tricky but it simplifies the computation !

   If we would be at $t = t_1$ then we have two bonds, so let us do "as if" we are at $t = t_1$. Then, at that moment we would compute the swap rate as the fixed rate that makes difference of two bonds equal to zero:$B_{fix} - B_{float} = 0$ and we knew that this means that the value of the floating bond was the face value F at t_1. But we can not be at $t = t_1$ because that is in the future.

   So let us go back to $t = 0$, then we add zero cash flows at the start of the fix leg, and for the floating leg we just discount the face value to $t = 0$.

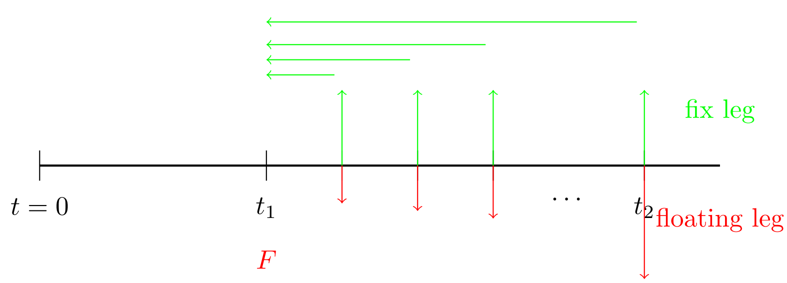


   Therefore we have to solve $F \times DF(0, t_1) = \sum_{t=\Delta t}^{t_2} F \cdot S_{t_1,t_2} \cdot \Delta t \cdot DF(0, t) + F \cdot DF(0, t_2) = 0$ for S_{t_1,t_2}.

   We find that $F \cdot S_{t_1,t_2} \cdot \sum_{t=\Delta t}^{t_2} \Delta t \cdot DF(0, t) = F \times DF(0, t_1) - F \cdot DF(0, t_2)$ or

$$S_{t_1,t_2} = \frac{DF(0, t_1) - DF(0, t_2)}{\sum_{t=\Delta t}^{t_2} \Delta t \cdot DF(0, t)}$$

   To clearly show the difference with a usual swap we show below the figure assuming that we are at t_1 !



At $t = t_1$ you would solve $F = \sum_{t=\Delta t}^{t_2} F \cdot S_{t_1,t_2} \cdot \Delta t \cdot DF(t_1, t) + F \cdot DF(t_1, t_2) = 0$ for S_{t_1,t_2}

Remark 3.8. *As for an FRA, for some time point t between $t = 0$ and $t = t_1$ all values have changed and this forward starting swap will have a value different from zero. At initiation its swap rate is fixed such that the value at initiation is zero ! Therafter it can have a non-zero value.*

To compute that non-zero value you have to apply "method of the cash-flows" or the "method of the series FRAs"

3.8 NEW: Excell sheets seen in the course.

3.8.1 Index rate swap on slide 31

Important notes:

Daycount convention: Note the daycount convention in cell J3. This has to be taken into account wherever we use t or Δt. In this case it is 30/360 which means that one month is 1/12, so when we work in multiple of month there is nothing special. This would be more complex if actual where used in the numerator or denominator !

Computation of 'Zero Rate': The term structure of the interest rates is given as a Nelson-Siegel curve with parameters given in the excel, namely $\alpha_1 = -0.01, \alpha_2 = -0.005, \alpha_3 = 0.5, \alpha_4 = 0.035$. (these are a1,a2,a3,a4 in $B4 : B7$ and the values in $D4 : D7$).

As we have seen (see chapter 1) this means that the zero coupon rates can be found as $(\alpha_1 + \alpha_2 t)e^{-\alpha_3 t} + \alpha_4$, this is the formula you find in the cells $D13 : D33$. The time t is in the columns $B13 : B33$

Computation of 'DF': From the zero rates yuo can compute the discount factors (see overview on interest rate swaps) using $DF(0, t) = e^{-r(t-t_0)}$, where r is the zero rate, t is in column B and t_0 is the first t that you have so $t_0 = 0$. we have continuous compounding in discount rates, which is usual. . This is the formula you find in $E13 : E33$.

Computation of the 'Fixed cash flow': this is just the fixed rate (given in H5) times the nominal amount (in H3) times $\delta t = 0.5$. This is the formula in G13:G33. he might ask something with above or below par value ?

Computation of the 'Floating rates': These are (see supra when we computed interest rate swaps) derived from the discount factors with the formulas explained in "overview" supra. Note, we have FRA here so no compounding is used !!! These are the formulas in H13:H33.

Computation of "Floating cash flow": similar to fixed cash flow but with the floating rate. Formulas in I13:I33.

Computation of "FRA Value": difference of the receive - pay, discounted. Formulas in J13:J33

Computation of cell 'I9': this is the sum of the discounted floating cash flows, so floating casf flow times discount factor and then sum.

Computation of cell 'J9': sum of FRA values (they are already discounted, see supra)

Computation of cell 'Check': this is the sum of the discounted fixed cash flows, this should be equal to receive - pay (I9-J9).

3.8.2 NEW: Cross Currency swap on slide 48.

TO DO

4 Properties of options

see Hull and slides, study these slides, for completeness I have just commented on some of the slides:

slide 4: Bermuda is an island between Europe and America, so A bermudan option is somewhere between a European option (only exercised at maturity) and an American option (can be exercised at any time until its maturity).

A Bermudan option is an American option that can be exercised at a fixed set of dates before its maturity.

slide 6: intrinsic value, in the money, ...

slide 7: option premium

slide 18: understand graph :

We found that the value of a European call with exercise price K on an underlying stock without dividends is bounded below by the stoc price and bounded above by the present value of the exercise price:

$$S - Ke^{-r(T-t)} \leq c(S) \leq S$$

If we draw the line $c(S) = S$, then for $S = 0$, $c(S) = S$ is also zero because $c(S) = S$ and S was zero. Similar, in $S = 1$ we find $c(S) = 1$ so the line c(S) is the line through the origin and with slope 1.

The line $c(S) = S - Ke^{-r(T-t)}$ also has a slope of one, so it is parallel to the line above. For $S = Ke^{-r(T-t)}$ it holds that $c(S) = S - Ke^{-r(T-t)} = 0$ so the intersection with the horizontal axis is $Ke^{-r(T-t)}$. Note that this latter value is smaller than K because $e^{-r(T-t)}$ is smaller than or equal to one.

As such we have three regions:

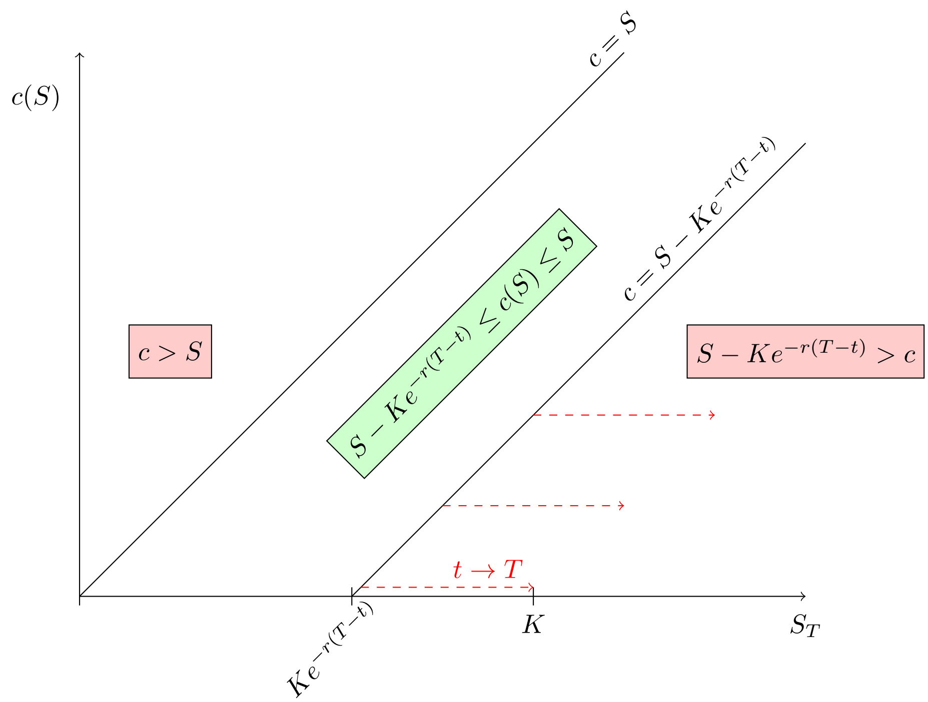
Between the vertical axis and the line $c = S$: take an "easy point" in this region, e.g. on the vertical axis. For such point on the vertical axis we see that S is zero while c is positive, so in this region $c > S$

Between the line $c = S$ and the line $c = S - Ke^{-r(T-t)}$:

At the right of $c(S) = S - Ke^{-r(T-t)}$:

It is important to note that the boundary $S - Ke^{-r(T-t)}$ at the right changes in time !, at T this boundary moves toward K ! This is indicated by the red, dashed arrow

Figure 3: European long call, no dividends, same for Americal long call !



slide 19: Question on American call option without dividend, answer it !

Assume at $t < T$ the American option is deep in the money (i.e. the current stock price S_t is much higher than the strike price K in the option), will you exercise it? you have a long position in the call, else you do not have the choice but you have to undergo the choice when you are short.

- If you exercise: then you pay K today at t and you will own the stock until T
- If you do not exercise: then you pay K at T and you own the stock at T

So at T you will own the stock in both cases, but in the first case you have paid K at t and in the second case you paid K at T. Since the value of money is higher when t is earlier, you have paid more in the first case ! so you will not exercise an American call early (when the underlying has no dividends). With dividends this can be different !

Moreover, it could be that, even though at t $S_t > K$, at T the stock price has fallen such that it could be that $S_T < K$ and then it is cheaper to buy at S_T.

As an American call on a stock with no dividends is never exercised early, this implies that the graph with the boundaries is the same as for a European call without dividends, so the same as on slide 18 !

Take this into account in exercises !

slide 22: **NEW:** understand the graph ! We found that the value of a European put with exercise price K on an underlying stock without dividends is bounded below by the stoc price and bounded above by the present value of the exercise price:

$$Ke^{-r(T-t)} - S \leq p(S) \leq Ke^{-r(T-t)}$$

If we draw the line $p(S) = Ke^{-r(T-t)}$, then as there is no S in the equation, it is a horizontal line (derivative toward S is zero). The line $p(S) = Ke^{-r(T-t)} - S$, cuts the vertical axis in $Ke^{-r(T-t)}$ (put $S = 0$, because on the vertical axis S is zero, to see that) and has slope -1.

As such we have three regions:

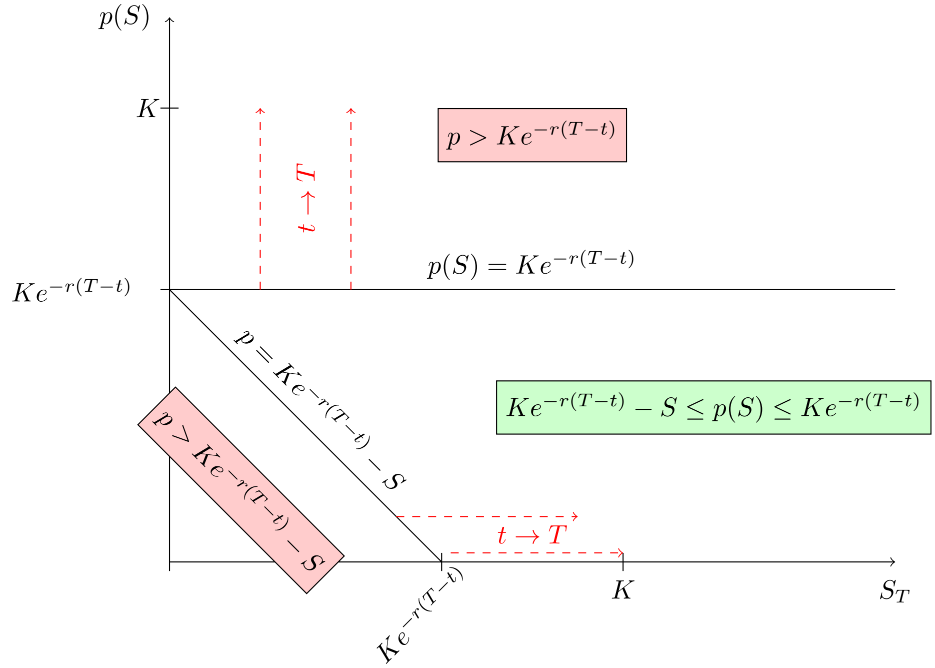
Between the vertical axis and the line $p = Ke^{-r(T-t)} - S$**:**

Above the horizontal line $p = Ke^{-r(T-t)}$**:**

At the right of $p = Ke^{-r(T-t)} - S$ **and below** $p = Ke^{-r(T-t)}$**:** This is the zone of the possible prices !

It is important to note that the horizontal boundary $p = Ke^{-r(T-t)}$ changes in time , just as well as !, at T this boundary moves toward K ! This is indicated by the red, dashed arrows

Figure 4: European long put, no dividends



slide 23: early exercise of put can be profitable even without dividends !

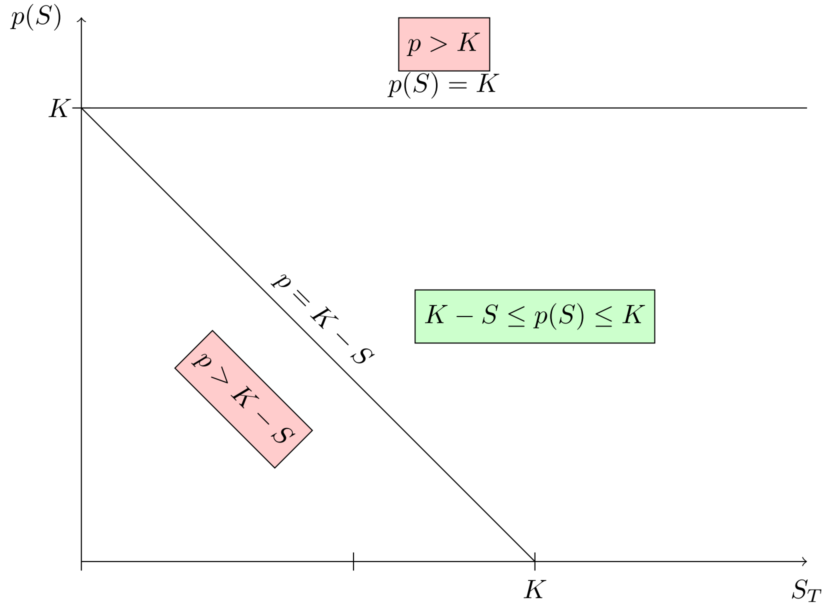
Assume at $t < T$ the American (long) put option is deep in the money (i.e. the current stock price S_t is much lower than the strike price K in the option), will you exercise it? (Note that it should be a long put because with a short put you have no choice to make, you must undergo with a short put !)

- If you exercise: then you receive K today, and deliver the stock today (with a value of $S_t << K$); i.e. you get $K - S_t$ at t
- If you do not exercise: then you receive K at T and deliver the stock (valued S_T), i.e. you get $K - S_T$ at T
  So if $K - S_t > (K - S_T)e^{-r(T-t)}$ then it is advantages to exercise the American put early. Tis is the case if S_t is very low or if r is very high, so there are occasions when this may happen !

This means that, in the picture with the boundary values of the put option premium, we can receive K even before T, namely if we decide to exercise the long put at t that is before T, then we receive K at t. So the figure for an American long put is different !

NEW:

Figure 5: American long put, no dividends, at any $t < T$



slide 25-29: know impact of dividends.

slide 28: Impact of dividend payments on an American call:

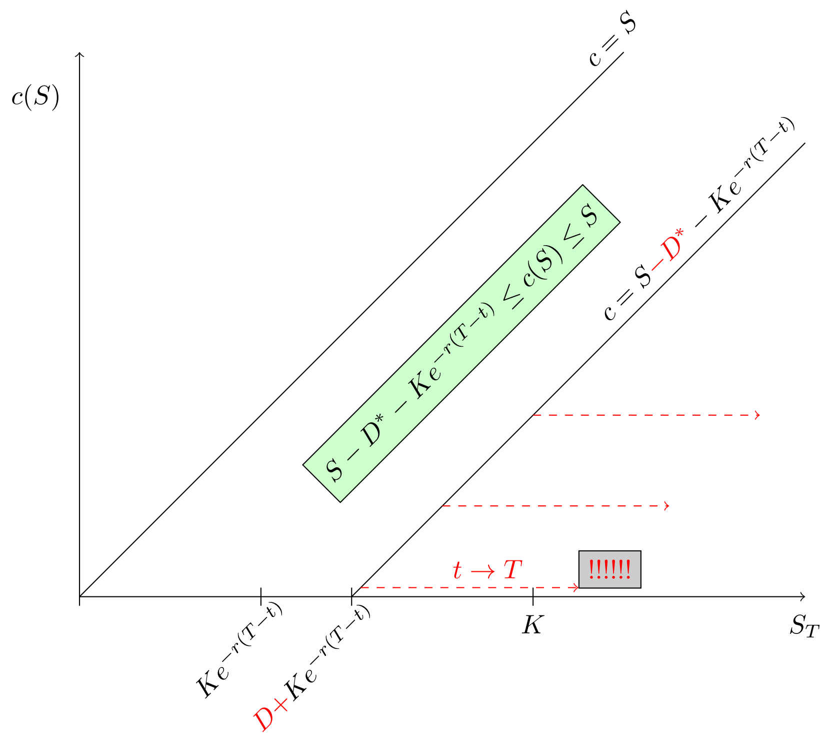
The effects of dividend payments can easily be analysed. As a rule of thumb in the formulas for the boundaries you you replace S by $S - D^*$ where D^* is the sum of the discounted dividends. Compare this to a forward without S_0 and with $(S_0 - D^*)$ dividends.

So for the call option e.g. the lower boundary becomes $c(S) \geq S - D^* - Ke^{-r(T-t)}$.

Therefor the slide for a call option will no longer be the one on slide 18. The line that delineates the region on the right will intersect the horizontal axis at $D^* + Ke^{-r(T-t)}$:

As indicated in the graph below this has very important implications; the right boundary can go further than K !!! (because of D^*), even for times t that are before maturity. If we look at the arguments to illustrate that an American call without dividends will never be exercised early, then we see that these arguments no longer hold. So for a dividend paying stock an American call may be exercised early !
This not a surprise because if you have the right to exercise it at any time, you may exercise it and obtain the stock, just before the date that a dividend is paid !!!
Please read slides, he does this in terms of intrinsic value !
Compare this to the payoff profile of a European call !

Figure 6: European and American long call, with dividends



slide 29 NEW:: Impact of dividend payments on an American put:

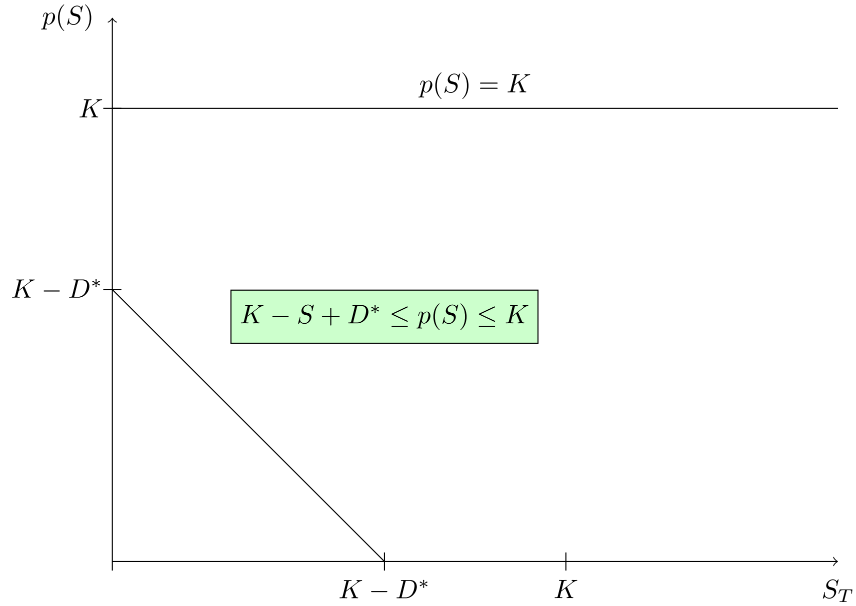
The effects of dividend payments can easily be analysed. As a rule of thumb in the formulas for the boundaries you you replace S by $S - D^*$ where D^* is the sum of the discounted dividends.

So for the call option e.g. the lower boundary becomes $Ke^{-r(T-t)} - (S - D^*) \leq p(S)$.

Therefore the slide for a put option will no longer be the one on slide 19. The line that delineates the region on the right will intersect the horizontal axis at $Ke^{-r(T-t)} - D^*$:

As indicated in the graph below this has very important implications; the right boundary is shifted more to the left by the $-D^*$.
So if we include this in the American put then we get the figure below. Dividends make early exercise less profitable, this was expected because as log as you do not exercise the put, you own the underlying stock and you receive all the dividends, once you have exercised the put, you lose ownershiwp and all the future dividends ! So if you want to exercise early, always wait until after a dividend date !
Clearly indicate that D^* is the (sum of) the present values of the dividends, the professor uses $PV(D)$.

Figure 7: American long put, with dividends



slide 30-42: put-call parity

slide 50: II think this is linked to one of his exam questions, so we have to look at it

Slide 81: Slide 81 is particular because of the drop in the payoff. you will need a binary option for that.

Definition 20. *A long call has a payoff of* $max(S_T - K, 0)$ *meaning that, if* $S_T > K$ *you receive* $S_T - K$ *and if* $S_T \leq K$ *then you receive zero.*

The binary option looks like that, but you get a fixed amount, in other words, a binary long call has a payoff if $S_T > K$ you receive 1 and if $S_T \leq K$ then you receive zero

A binary short call has a payoff if $S_T > K$ you receive -1 and if $S_T \leq K$ then you receive zero

	$S_T \leq 50$	$50 < S_T < 65$	$S_T = 65$	$65 < S_T < 70$	≥ 70
Target profile	0		+1		0
!!			drop -9		

So the slope profile is (0,1,0) but at 65 you additionaly need a drop so you short 9 binary calls with a strike of 65 (not sure about this, can you ask a question to the assitant ?.

Note that apart from that (0,1,0) is a bull spread.

zeker de put-call parity bekijken

5 Trading strategies with options

see Hull and slides

de logica om het te begrijpen staat onder 6.1, de verschillende strategies (butterfly, ...) ken je al en kan je afleiden met wat in 6.1 staat, dus lees eerst 6.1.

6 Options

6.1 Option strategies

6.1.1 Long/short positions in deriatives contracts.

forwards/futures: We have seen that a long position in a forward means that you are the buyer in a forward contract meaning that have an obligation to buy at the maturity date T. A short position in a forward means that you have an obligation to sell at T.

Options: Options are a bit more complicated because there are put and call options, but an options is not about an obligation but about a right. Your are long when you own the right at T and short when you have to undergo the right.

So being long in a call means that you have the right to buy the underlying at T, being being long in a put mans that you have the right to sell at T. Being short in a call means that you have to "undergo" the right to buy, so you must sell when the owner decides to buy.

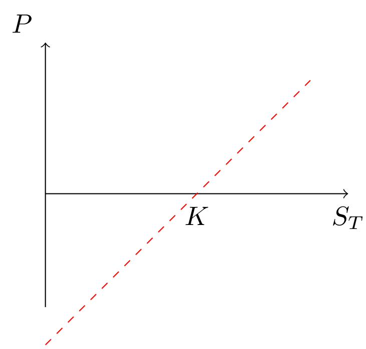
6.1.2 Payoff functions at maturity T

.

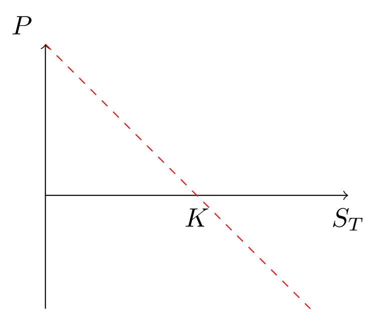
The payoff diagrams/functions of a position in a derivative learn us what the profit is of being in that position as a function of the stock price at maturity.

Forward/future: Let's look at a long/short position:

- Long position in a forward: this means that at T I have the obligation to buy the underlying at the contract price K. If K, the contract price, is smaller than the spot price at that moment, then I have a profit of $S_T - K$, if K is more than the spot price then I lose $K - S_T$, so I have a negative profit of $S_T - K$. If S_T is on the horizontal axis, then $S_T - K$ is a line with s lope of 1, if $S_T = 0$ then the line crosses the vertical axis at $-K$, if $S_T = K$ then we are at zero on the vertical axis:



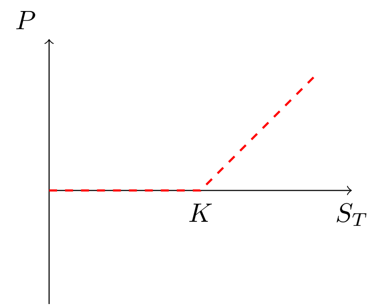
- Short position in a forward: this means that at T I have the obligation to sell the underlying at the contract price K. If K, the contract price, is smaller than the spot price at that moment, then I have a loss of $S_T - K$ or a profit of $K - S_T$, if K is more than the spot price then I win $K - S_T$. If S_T is on the horizontal axis, then $K - S_T$ is a line with s lope of -1, if $S_T = 0$ then the line crosses the vertical axis at K, if $S_T = K$ then we are at zero on the vertical axis:



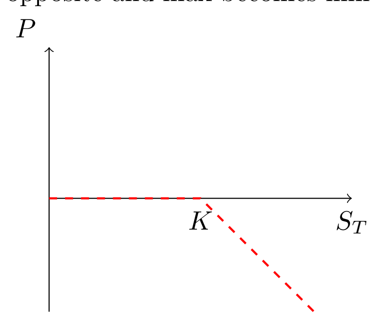
Options: We distinguish between a call option and a put option:

Call options: Let's analyse a short and a long position in a call option:

- A long call position mean that we have the right to buy at T, so if we want we can by at K, we will do that is S_T is higher than K but if S_T is lower then we will buy on the spot market because that is cheaper than K. So if $S_T < K$ then we wil not earn from our ritght ro buy, else we earn $S_T - K$, so if S_T is on the horizontal axis, then left of K (where $S_T < K$) our profit from holding the option is zero) and right of K it is $S_T - K$, a half-line with slope 1.
  Note that left of K it holds that $S_T - K < 0$ and then $max(S_T - K, 0) = 0$, and right of K it holds that $S_T - K > 0$ so $max(S_T - K, 0) = S_T - K$, so the payoff is also equal to $max(S_T - K, 0)$.

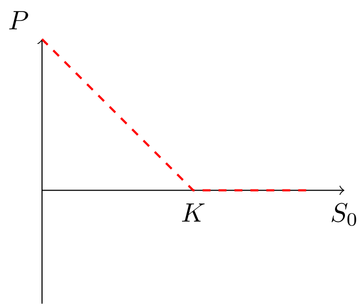


- A short call is the opposite, so we find the payoff diagram below. Note then when you multiply by '-1' then inequalities turn to the opposite and max becomes min, so the payoff is $min(K - S_T, 0)$

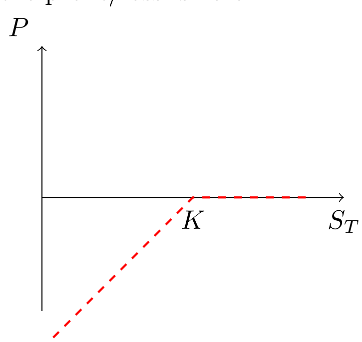


Put options: Let's analyse a short and a long position in a put option:

- A long put position means that we have the right to sell at T at a price K,



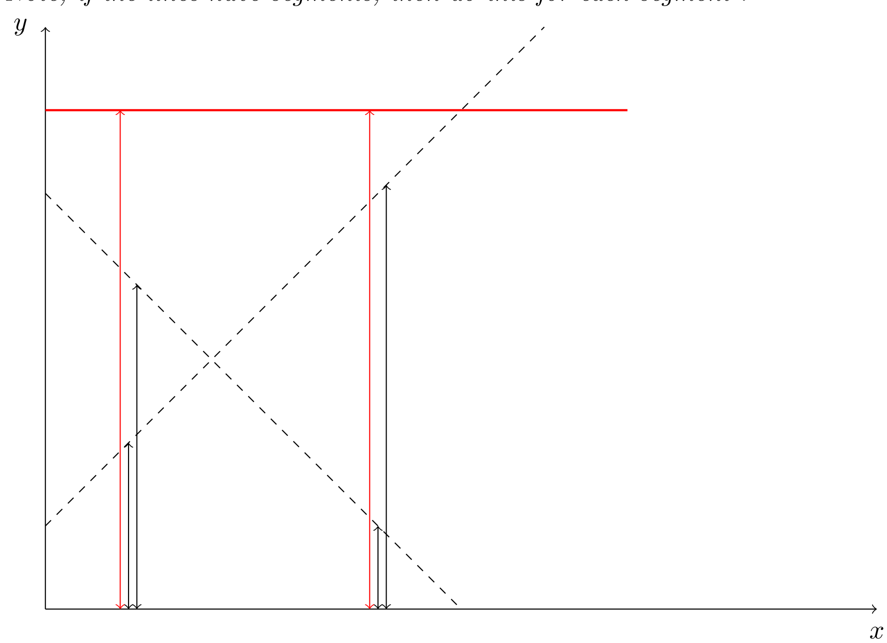
- A short put is the opposite, we have to undergo the right exercised by the owner of the put; so if $S_T < K$ then the owner will exercise his right and sell to you at K while we could have bought at $S_T < K$ in the spot (at T), so you loose $K - S_T$ or you win $S_T - K$ which is a line with an upward slope. After K the profit/loss is zero



We summarize this in the payoff table below:

Derivative	comment	$S_T < K$	$S_T = K$	$S_T > K$	summary
long forward	must buy at K	loss $S_T - K(< 0)$	0	win $S_T - K(> 0)$	$P(S_T) = S_T - K$
short forward	must sell at K	win $K - S_T(> 0)$	0	loss $K - S_T(> 0)$	$P(S_T) = K - S_T$
long call	right to buy at K	buy spot, 0	0	exercise $S_T - K(> 0)$	$P(S_T) = max(S_T - K, 0)$
short call	undergo	0	0	$K - S_T$	$P(S_T) = min(K - S_T, 0)$
long put	right to sell at K	win $K - S_T$	0	0	$P(S_T) = max(K - S_T, 0)$
short put	undergo	lose $K - S_T$	0	0	$P(S_T) = min(S_T - K, 0)$

Remark 6.1. *Adding up graphs:*
Choose a point in the horizontal axis and measure the distance to the first line, for that same point on the horizontal axis, measure the distance to the second line. This is illustrated by the two black arrows at the left. Add up these two distances and take this is the vertical highet above the point on the x-axis (red line at the left).
Do the same for a second point. Now you have tow points on the line that is the sum of the graphs (the red yhick line).
Note, if the lines have segments, then do this for each segment !

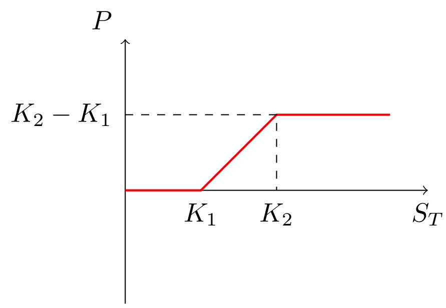


6.1.3 Option strategies.

Bull spread: The figure of the **Bull spread** is illustrated below.

A bull spread stragey looks as in the picture below. It consists of a horizintal line (this is a line with slope zero), followed by a line with slope +1, followed by a horizontal line i.e. slope zero.

We have to find basic options/futures positions that give this pattern. The descriptions of these basic payoffs as a grapgh and as a combination of slopes is given on slide 47. E.g. a long call starts horizontally (slope zero) and rises from its strike price on with a slope +1. So it has slope zero followed by slope 1.



If we loop at the bull spread then we have a sequence of three slopes, horizontal (slope 0), from K_1 on it is rising at slope 1 and then from K_2 it is again horizontal with slope 0.

So our target profile, in terms of slopes is :

	$\leq K_1$	between K_1 and K_2	$> K_2$
Target profile	0	+1	0

How can we find basic positions that combine to this figure ? Start looking at the figure for the bull spread, at the left hand side and find a naked (or basic) strategy like that. If you look at slide 47 then you see that the begin (at the left) looks as Long call with strike price K_1. So a long call with strike price K_1 has a "slope profile" ginevn in the table below:

	$\leq K_1$	between K_1 and K_2	$> K_2$
Long call, strike K_1	0	+1	+1

So it is fine at the beginning, but after K_2 it does not fit our target. To come to our target, we may make no change to the slope until K_2, so we have to add a something with slope zero until K_2 and after K_2 we have to add something that compensates the slope in the above table (+1) to the one in our target table (0). So we need something with slope zero until K_2 and then -1 after K_2. The naked psoitions at slide 47 learn that we need a short call with strike price K_2. Let's try:

	$\leq K_1$	between K_1 and K_2	$> K_2$
Long call, strike K_1	0	+1	+1
Short call, strike K_2	0	0	-1
Combination (sum)	0	+1	0

If you compare this to our target profile, then we see that the target is obtained by combining a long call with strike price K_1 and a short call with strike price K_2.

There are other solutions, with long calls, but it's a bit more tricky:

Note that I need a slope sequence (0,1,0) and supra this was found as (0,1,1)+(0,0,-1)=(0,1,0). But I could also add other combinations:

(-1,0,0)+(1,1,0)=(0,1,0) so if I add up two basic (naked) strategies (-1,0,0) i.e. negative slope followed by horizontal from K_1 on and (1,1,0) i.e. positive slope and from K_2 on horizontal, then I also get the target !

These are a long put with struke K_1 (negative slope followed by horizontal from K_1) and s sohort put woth strike K_2 (positive slope and from K_2 on horizontal)

Bear spread: can be constructed in two ways:

6.2 The value of a call option; replicating portfolio's

Let's analyse a share with spot price S_0 at $t = 0$ and let us assume (hence 'tree') that the price can go up in the next period to $S_0 \times u$ or go down to $S_0 \times d, d < u$[9].

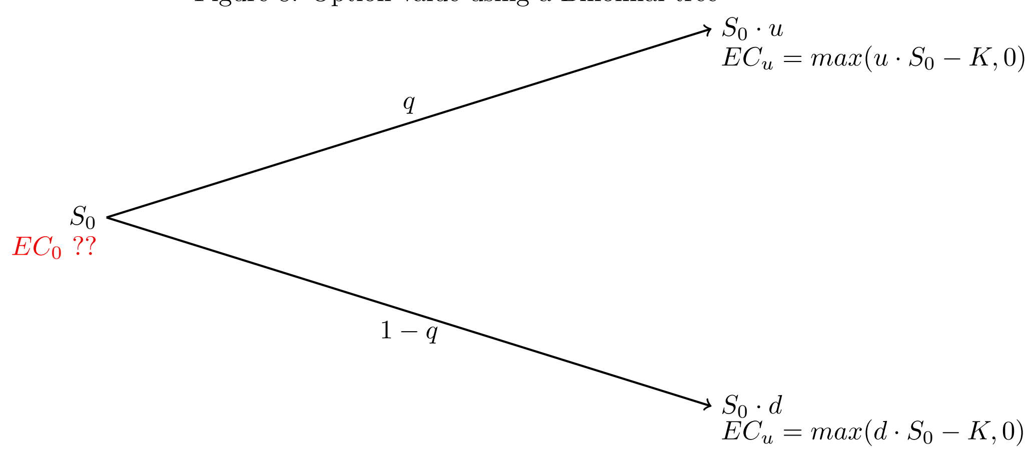
Then at $t = 1$ the value of a European call with exercise price K will be in the first case $EC_u = max(u \cdot S_0 - K, 0)$, in the second case it would be $EC_d = max(d \cdot S_0 - K, 0)$. This is shown in figure 8.

q is the probability that the price moves up, consequently $(1 - q)$ is the probability of a downward move.

We would like to find the value EC_0 of a call option today.

Note that at $t = 0$ the value of S_0 can be observed on the spot market, so we know the value of S_0, with that (and other information) we wish to find the value of a call option at $t = 0$. Note for a put it is still different !

Figure 8: Option value using a Binomial tree



In order to do this, we will create a **"replicating portfolio"** also called a **"synthetic option"** as follows: we will borrow money (an amount B) and buy the stock Δ shares, and at the same time sell European calls.

Remark: This Δ is just a variable, it is confusing because Δ usually has the meaning of "difference" but here it is not, it would have been better to use e.g. N.

[9] so if the price increases by 10pct then $u = 1.1$.

Remark 6.2. *Note that this is similar with what we dit for a forward*[10]*: we borrowed money to buy the stock and because of arbitrage we found (simple case)* $F_0 = S_0 e^{r\Delta t}$.

Because of the assymetry in options you will not have to buy one share but "an unknown number of shares".

Remark 6.3. *This does not work for a put option, with a call we can buy stock today and keep it, or we can buy it at* T*. That does not work for a put !!!, we will have to use the put-call parity to find the value for a put (CHECK).*

Moreover, we will make that portfolio to "replicate" the option, that means that, whatever the the price movement is, the value of our portfolio is the same as the value of the option, so if the price moves to $S_0 u$ then our potfolio must be woth EC_u but if it moves to $S_0 d$ the value of our portfolio must be EC_d. That is the case if both of these conditions are fulfilled:

$$\begin{aligned} \Delta \cdot S_0 \cdot u + B \cdot e^{r \cdot \Delta t} &= EC_u \\ \Delta \cdot S_0 \cdot d + B \cdot e^{r \cdot \Delta t} &= EC_d \end{aligned}$$

Subtracting both equations we find $\Delta \cdot (S_0 u - S_0 d) = EC_u - EC_d$ or $\Delta = \frac{EC_u - EC_d}{(S_0 u - S_0 d)}$ and subsituting this in one of the other equations you find $B = \frac{u \cdot EC_d - d \cdot EC_u}{(u-d)e^{r\Delta t}}$.

$$\begin{aligned} \Delta &= \frac{EC_u - EC_d}{(S_0 u - S_0 d)} \\ B &= \frac{u \cdot EC_d - d \cdot EC_u}{(u-d)e^{r\Delta t}} \end{aligned}$$

So if we know $u, d, S_0, r, \Delta t$ then we can compute Δ and B such that a portfolio of Δ shares of the stoch and a loan of B EUR have the same value as a call option after one period. In other words the value today should also be the value of that protfolio today and we find that

$$EC_0 = \Delta \cdot S_0 + B \tag{28}$$

where Δ and B are as supra.

So we can now find a value for the call option, if we know $u, d, S_0, r, \Delta t$. At $t = 0$ we know S_0 and r, we also know Δt. u and d are problematic, we can not know the value of the share in the future ! We will see how we can find these later.

Remark 6.4. *Note that* $\Delta = \frac{EC_u - EC_d}{(S_0 u - S_0 d)}$ *and I think (CHECK) that this* $\Delta \leq 1$ *and in most cases strictly smaller. Remember that for a forward (see section*

[10]Note that this is the same reasoning as for determining the delivery price for a forward, see section 2.3.1. Only because we are not sure that the option will be exercised we can not buy a full unit of the stock, but a number Δ of stock.

2.3.1) we bought a full share of stock at $t = 0$, for an option we only buy a part (Δ), that is because at maturity you are not sure to be able to buy it, it is an option, so sometimes it will be sometimes not and "on average" it can be less than 1 share that is bought at maturity !

First we re-write the value EC_0 using the expressions for Δ and B:

$$
\begin{aligned}
EC_0 &= \frac{EC_u - EC_d}{(S_0 u - S_0 d)} S_0 + \frac{u \cdot EC_d - d \cdot EC_u}{(u-d)e^{r\Delta t}} \\
&= \frac{EC_u - EC_d}{(u-d)} + \frac{u \cdot EC_d - d \cdot EC_u}{(u-d)e^{r\Delta t}} \\
&= \frac{EC_u - EC_d + u \cdot EC_d \cdot e^{-r\Delta t} - d \cdot EC_u \cdot e^{-r\Delta t}}{u-d} \\
&= \frac{EC_u(1 - d \cdot e^{-r\Delta t}) + EC_d(u \cdot e^{-r\Delta t} - 1)}{u-d} \\
&= e^{-r\Delta t}\left(EC_u \frac{e^{r\Delta t} - d}{u-d} + EC_d \frac{u - e^{r\Delta t}}{u-d} \right)
\end{aligned}
$$

6.3 Risk neutral probabilities.

Note that $\frac{e^{r\Delta t}-d}{u-d} + \frac{u-e^{r\Delta t}}{u-d} = 1$ so if we define $p = \frac{e^{r\Delta t}-d}{u-d}$ then this becomes:

$$EC_0 = (p \times EC_u + (1-p)EC_d)\, e^{-r\Delta t} \tag{29}$$

If we treat p as a probability to move to $S_0 \cdot u$ (remember from figure 8 that the true (unknown probability is q)) then (there are only two possibilities) $(1-p)$ can be seen as the "probability" to move to $S_0 \cdot d$ and it can be seen from this equation that[11].

Property 25. *The value of the call at $t = 0$ i.e. EC_0 is the expected value of the payoff of the option in Δt discounted to $t = 0$ but therefore we have to work with a redefined probability p in stead of the "true" probability q.*

This "new probability is "as if" we have changed the unit of measurement for probabilities, and therefore we do not get no longer the value q that we observe in the real world (see figure 8) but because of the "change of units" we get another value $p = \frac{e^{r\Delta t}-d}{u-d}$.

Moreover, when we use this "new probability measure" we can discount at the risk free rate of interest, even for a call option which is usually not a risk free instrument. In other words, if we change from our "real world" where the probability of a price increase is q, to a new world where probabilities are measured in "another unit" then the value of an option can be computed

[11]Remember that the expected value of a random variable is the sum of each outcome of the variable, multiplied by the probability of the outcome, i.e. $\mathbb{E}(X) = \sum_i p_i x_i$ in this case we have to outcomes EC_u and EC_d each with an "artificially constructed" probability p resp. $(1-p)$.

very easily nemaly as the expected value of the option price for the next period, discounted at the risk free discount rate. So this world is as if it is "risk neutral".

Property 26. *So in the risk neutral world the computation of call options becomes very easy, (1) we take the expected value of the payoffs in the next period, using the risk neutral probability and (2) discounting is at the risk free rate. Note however that probabilities is this world have different values than these in our real world !*

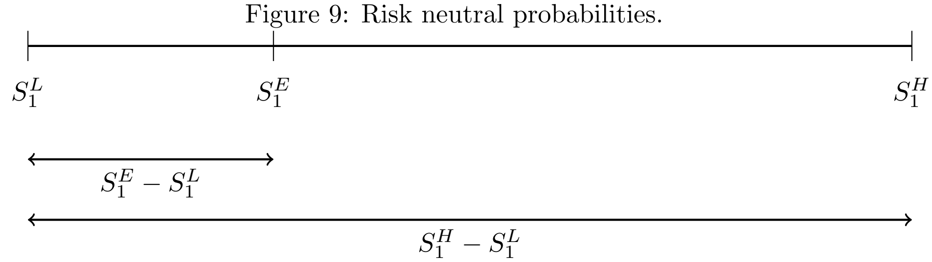
The risk neutral probability of an upward price move is defined as

$$p = \frac{e^{r\Delta t} - d}{u - d}$$

The "clue" underlying the risk neutral world is that we value an options in terms of the underlying stock, so that the formulas stay the same when the stock price changes (because the formulas are in terms of these stock prices).

Note that $p = \frac{e^{r\Delta t}-d}{u-d} = \frac{S_0 e^{r\Delta t} - S_0 d}{S_0 u - S_0 d}$, remember that (in the simples case) $F_0 = S_0 e^{e\Delta t}$ is what we should take as the forward price, i.e. out "estimate" of the price in the next period $t + \Delta t$. Let's note this as S_1^E, further $S_0 u$ is the high price in the next period, so let's note it S_1^H and similar S_1^L for the low price in the next period, then it can be seen that $p = \frac{S_1^E - S_1^L}{S_1^H - S_1^L}$. These values are schematised in the figure 9.

Figure 9: Risk neutral probabilities.



Property 27. *So the risk neutral probability measures the position of the "estimated" price in the next period, relative to the difference between the highest and the lowest price in the next period.*

So the risk neutral probability measures probabilities in terms of prices of the underlying stock !

6.4 How to get values for u and d

You should not know this in detail, but it will help you to understand the formulas and therefore it is very usefull to make all kinds of exercises.

Note that $u \cdot S_0$ is the high price in the next period $t + \Delta t$, let us note it as $S_{\Delta t} = S_0 \cdot u$. Then the difference in price between the two periods is

$\Delta S = S_{\Delta t} - S_0 = S_0 \cdot u - S_0$ and the relative price change is $\frac{S_{\Delta t}}{S_0} = u - 1$. So $u - 1$ is the relative change for the highest price in the next period. Similar for $d - 1$.

So u and d have to do with changes in the stock price, how can we "model"' these ?

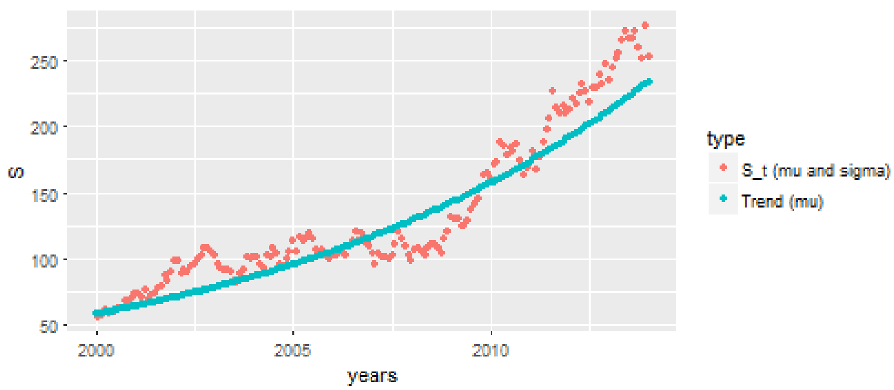
Price changes for a stock are a sequence of values $S_0, S_1, S_2, \ldots$, and we could try to finc some regression to fit a line, i.e. something like $S = \beta_0 + \beta_1 t + \epsilon$ where ϵ is the error term that has a normal distribution. It seemed that such a regression is too simple because the stock price S_t does not change in a linear way. It turns out that it is better to do a regression for $\frac{\Delta S}{S_0}$ as a function of Δt, i.e.

$$\frac{\Delta S}{S} = \mu \Delta t + \epsilon$$

where $\epsilon \sim N(0, \sigma\sqrt{\Delta t})$ or written differently, for each time point t, the $\frac{\Delta S}{S}$ is a normal distribution:

$$\frac{\Delta S}{S} \sim N\left(\mu \Delta t, \sigma\sqrt{\Delta t}\right)$$

Graphically this means that these stock prices look like in the figure[12]



This can help us to find values for u and d: we have a "model" where the price can go "up" or "down", the probability to go up is q, the probability to go down is $1 - q$. So this is a Bernoulli random variable.

So we have a growth $\Delta S/S = u - 1$ with a probability q and growth $\Delta S/S = d - 1$ with a probability $1 - q$, so the "expected growth" with this Bernoulli (or Binomial) model is $q \times (u - 1) + (1 - q)(d - 1)$. If the model supra, with the normal variable, is correct, then this should be the same mean or : $q \times (u - 1) + (1 - q)(d - 1) = \mu \Delta t$. For the variance we get a similar equation: $q(u - 1 - \mu\Delta t)^2 + (1 - q)(d - 1 - \mu\Delta t)^2 = \sigma^2 \Delta t$

[12] Indeed if $\frac{S_t - S_{t-1}}{S_{t-1}} = \mu\Delta t + \epsilon$, then $S_t - S_{t-1} = (\mu\Delta t + \epsilon)S_{t-1}$ or $S_t = (1 + \mu\Delta t + \epsilon)S_{t-1}$, if we use this result to replace $S_{t-1]}$, ... then you find $S_t = (1 + \mu\Delta t + \epsilon)^t S_0$. If ϵ would be zero then you have someting like compounding, this is the green line. The random ϵ gives the red line that deviates in an unpredictable (random) way from the green one.

We now have two equations with unknowns u, d, q, so we need one addional equation, we assume that $u \cdot d = 1$ or that $d = 1/u$ which means that the "up" or "down" movement are symmetric. (if I go up by u-1=10% then you go down by d-1=-10%, $u \cdot d = 1.1 \times 0.9 \approx 1$).

So now we have

$$\begin{aligned} q \times (u-1) + (1-q)(d-1) &= \mu\Delta t \\ q(u-1-\mu\Delta t)^2 + (1-q)(d-1-\mu\Delta t)^2 &= \sigma^2\Delta t \\ u \cdot d &= 1 \end{aligned}$$

If you have σ and μ then we can solve this to find u, d, q and we find

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}} \\ q &= \frac{e^{\mu\Delta t} - d}{u - d}, \text{ "true world" probability} \end{aligned}$$

μ and σ can be found using regression techniques.

Remember that the risk neutral probability is

$$p = \frac{e^{r\Delta t} - d}{u - d}, \text{ "risk neutral" probability}$$

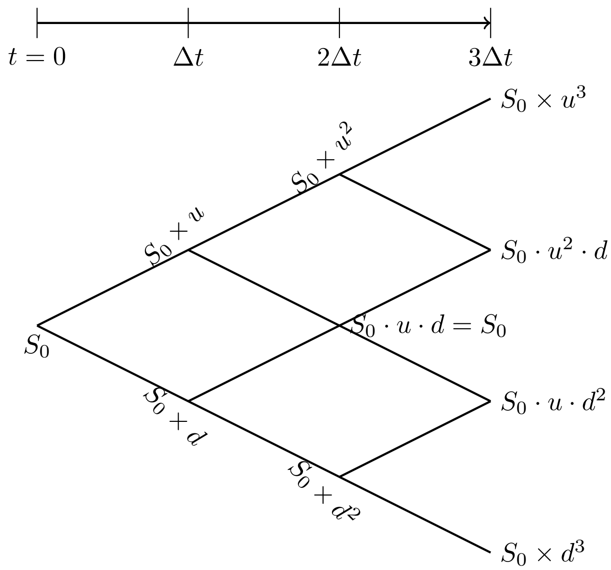
For a European call option (EC) without dividend payments it holds that:

$$EC_t = e^{-r\Delta t}\left(pEC_{t+\Delta t,u} + (1-p)EC_{t+\Delta t,d}\right)$$

where r is the risk free rate of interest, p the risk neutral probability as above, $EC_{t+\Delta t,u}$ the value of the option in $t + \Delta t$ in case of a price increase and $EC_{t+\Delta t,d}$ the value of the option in $t + \Delta t$ in case of a price decrease.

6.5 Binomial trees

The valuation technique described above was used for "one step", i.e. we look one period Δt in the future. In more realistic cases we hvae to look at more than one period in the future: $\Delta t, 2\Delta t, \ldots$. Note that Δt is always in years, so e.g. 1 month is 1/12. Schematically we would get a tree-structure like in this figure (for 3 steps in the future) :

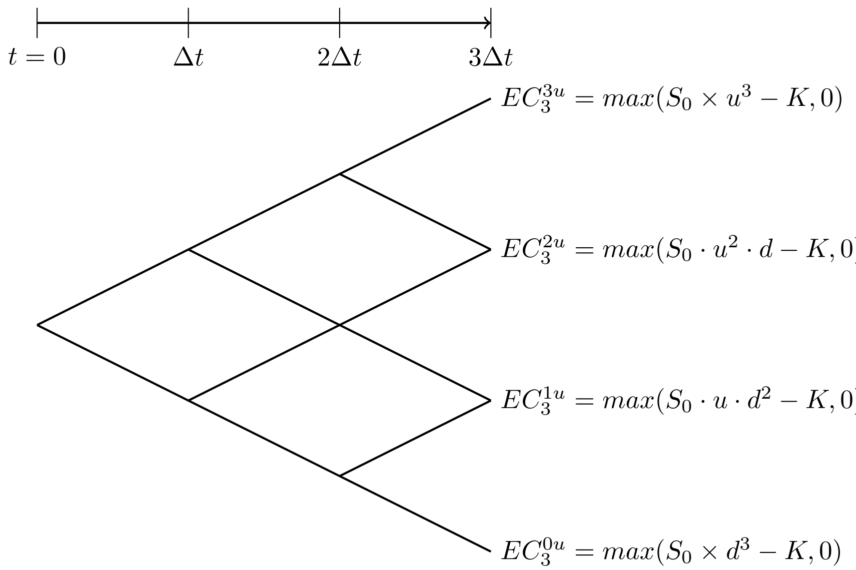


There are two ways to tackle that problem: backward computation and Binomial probabilities. Both methods are based on the finding that the value of a European call is the expected value of the future payoffs measured with the risk neutral probability p and discounted at the risk free interest rate.

Remember that a random variable has different outcomes x_i with each value having its own probability of occurrence p_i. The expected value of that random variable is then $\mathbb{E}(X) = \sum_i p_i x_i$. Note, in our case we have a value in case of an increasing price V^u with a probability p and a value down V^d with a probability $1-p$ therefore $\mathbb{E}(V) = p \cdot V^u + (1-p)V^d$.

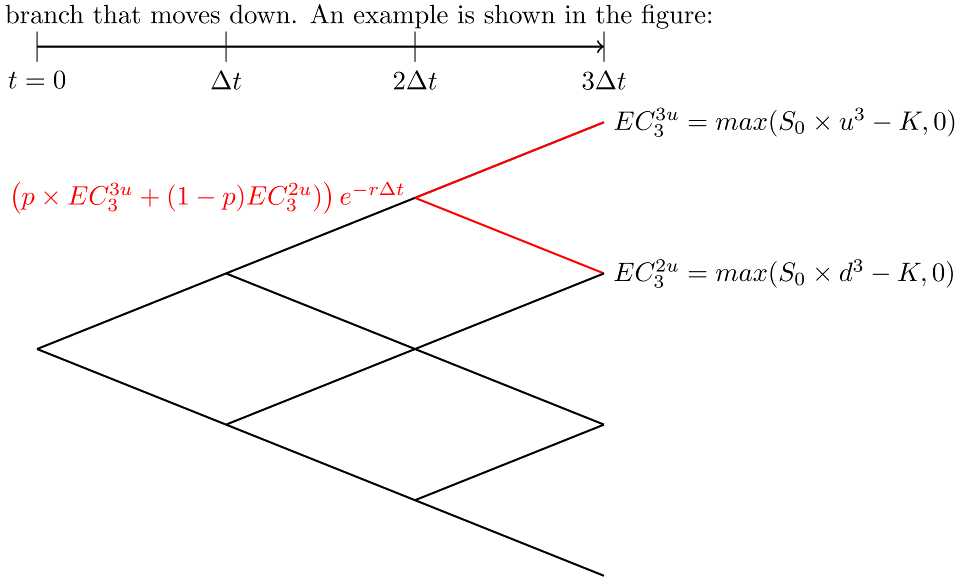
So in order to compute the expected value, we have to find (a) all possible outcomes and (b) for each outcome the probability of occurrence.

At maturity the option can have different values, depending on the spot price at maturity. For an exercise price K we have for a European call at maturity (the notation EC_3^{3u} means after three steps (subscript) that are three times up (3u)):



6.5.1 Backward computation

This method works "one step at a time" and from the right to the left: so move back, and at each intersecting node, compute the value in the node using the risk neutral probability p and the risk free interest rate r, i.e. $\left(p \times EC_3^{3u} + (1-p)EC_3^{2u}\right) e^{-r\Delta t}$. Note that p is always multiplied with the branch that moves up, $(1-p)$ with the branch that moves down. An example is shown in the figure:



Repeat this for all the nodes at $2\Delta t$. After that move backwards to Δt in a similar way and then at $t=0$ you find the value of the call.

6.5.2 Binomial probabilities

If the tree has a lot of steps then the backward computation will be very time consuming and error prone, the use of the Binomial probabilities will be much more efficient. This works as follows:

Here the reasoning is very similar, but we work only on the right side of the tree. At $3\Delta t$ we have the different possible values for the option at maturity. To compute the expected value we need also the probability of each outcome. We are now trying to find these probabilities.

- The probability to have a value of EC_3^{3u} after three steps is the probability to move 3 times up in three steps knowing that the probbaility to move up in one step is p. So I need (u,u,u) or three times u, the probability is threfore $p \times p \times p = p^3$.
- The probability to have a value of EC_3^{2u} is the probability to go two times up in three times, I can have (u,u,d) the probability is that we have u (probability p) another u (p^2) and d ($p^2(1-p)$). But we can also have (u,d,u) ($p(1-p)p$) and ((d,u,u)) ($(1-p)p^2$). Se we find $3p(1-p)^2$
- In general, if you have a three with N steps and you need to go up n times, knowing that in one step the probability to go up is p, you can compute the probability of n up moves in N steps as the Binomial probabilities $\binom{N}{n} p^n (1-p)^{N-n}$

So now we have, at maturity, all the possible values of out European call and the probability of each value. With this we can compute the expected value at maturity as the value times the probability and then sum all that:

$$\begin{aligned}
\mathbb{E}(V_{3\Delta t}) &= \overbrace{EC_3^{3u}}^{\text{value}} \times \overbrace{\binom{3}{3} p^3 (1-p)^{3-3}}^{\text{probability of value}} + \overbrace{EC_3^{2u}}^{\text{value}} \times \overbrace{\binom{3}{2} p^2 (1-p)^{3-2}}^{\text{probability of value}} \\
&+ \overbrace{EC_3^{1u}}^{\text{value}} \times \overbrace{\binom{3}{1} p^1 (1-p)^{3-1}}^{\text{probability of value}} + \overbrace{EC_3^{0u}}^{\text{value}} \times \overbrace{\binom{3}{0} p^0 (1-p)^{3-0}}^{\text{probability of value}}
\end{aligned}$$

This is the value at maturity, so we have to discount it at the risk free interest rate so our option value at $t = 0$ is

$$V_0 = \mathbb{E}(V_{3\Delta}) e^{-r(3\Delta t)}$$

6.6 The model of Black-Merton-Scholes.

6.6.1 The goal of this chapter.

In the previous section we have seen a simple model to find the value of an option that expires at time T in the future. This price depends on the values S_T in the future (see Binomial trees) and therefore;

1. We had to know all the values $S_{T,1}, S_{T,2}, S_{T,3}\dots$ of the underlying stock that are possible at T where T is in the future.
2. Compute the probabilities of each of the values for the stock p_i being the probability that the value of the stock at T is equal to $S_{T,i}$;
3. Compute the expected value of the option at T: $\mathbb{E}^{(p)}(EC_T) = \sum_i p_i \times max(S_{T,i} - K, 0)$; The superscript (p) is used to indicated that we changed to the "risk neutral" world, which is the same as the real world, but where probabilities are measured in other units.
4. Discount this result from T in the future to today: $EC_0 = e^{-r(T-0)}\mathbb{E}^{(p)}(EC_T)$.

We have seen how we do can this using a Binomial tree, however, we had to make simplifying assumptions: we assumed that we could proceed in several "steps" and that in each step the stock price can go up to S_0u or down to S_0d, so in the next step we can have only two different values.

NOTE: In reality we do not know the number of steps, and there will be much more possible values than these two, moreover we have assumed that u and d are the same in every step which is also unrealistic.

If u and p were the same all the time, then we are able to build a "replicating portfolio" with Δ shares of stock and a loan B. This number Δ depended (see supra) on numbers that are constant, so Δ was fixed between $t = 0$ and $t = T$

We need to generalise this to more realistic assumptions, so we have to find a more realistic "model" of how the price of the underlying stock evolves in the future.

- First we will generalise the number of steps, we will, between $t = 0$ and $t = T$ take an infinite number of infinitely small steps dt. So we will no longer work with sums, but with integrals;

- The second generalisation is that there are not only two possible values in the next step, but an infinite number that all are between $-\infty$ and $+\infty$. As before, we will also need the probabilities and we will assume that these are the probabilities of the normal distribution. So at each point in time between $t = 0$ and $t = T$, with steps of dt we will have a normal distribution for S_{t+dt}.

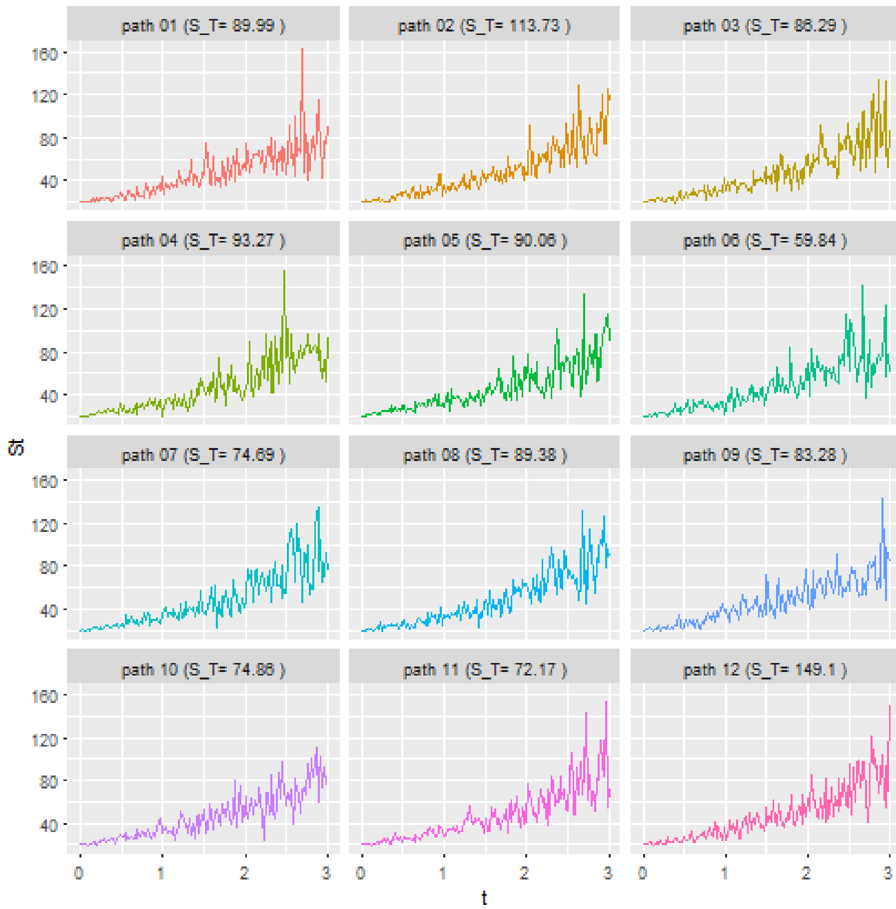
  Such a series of distributions is called a stochastic process, usually the distribution at at point in time t depends on the distribution in the previous time point (but not on the ones therefore), i.e. the Markov property.

We will see that we wil also contruct a "replicating portfolio", but the portfolio will change constantly in time !

We can do this for any t in the future (i.e. $t > 0$) so we have S_0 that is a single value, and S_{t_1} that is a distribution, S_{t_2} that is a distribution. So we have:

$$\overbrace{S_0}^{\text{known value}} \rightarrow \underbrace{\overbrace{S_{t_1}}^{\text{distribution}} \rightarrow \overbrace{S_{t_2}}^{\text{distribution}} \ldots \rightarrow \overbrace{S_T}^{\text{distribution}}}_{\text{Stochastic process}}$$

And each of these distributions is a normal distribution, this means that at each time point between $t = 0$ and $t = T$ we have a random outcome for the spot price S_t and thus also for S_T. Twelve examples of such outcomes are shown below:



You can see that at the end $t = T$ we find different values, that are reached via different paths. The problem that we face is that we have to find all the possible values for S_T and their probability, i.e. **how many of these paths bring us into that particular** value for S_T. For the Biniomial trees we worked with a simple model that made it such that these probabilites could be computed with a Binomial random variable using a risk neutral probability measure. when we work with our more general model, the stochastic process supra, this will be more complicated, but the results will be more realistic.

We will therefore first give a short summary on probabilities and probability distributions, in particular the normal distribution that we will use a lot.

6.6.2 Important properties of the normal distribution (repetition of stats course)

This is just a review of what you should remember from your statistics course:

A random variable is a variable whose outcome is not known in advance. What we do know however is (1) the list of all possible outcomes and (2) for each outcome we know its probability.

For example, rolling a dice can have outcomes 1 or 2 or ... 6, each with a probability of 1/6. But we do not know in advance what the outcome will be. For ach outcome x_i we know the probability of the outcome φ_i.

The expected value (or mean) of a random variable is defined as $\mathbb{E}(X) = \sum_i x_i \varphi_i$. Once we know the mean we can compute the variance $Var(X) = \sum_i \varphi_i (x_i - \mathbb{E}(X))^2$. The standard deviation is the square root of the variance: $\sigma(X) = \sqrt{Var(X)}$

These are definitions for discrete variables. For continuous variables the definition is similar only that we can not define the probability of a single value as outcome. We have to define the probability that the outcome is in an interval. Moreover the sums will become integrals. So if X is a continuous random variable with a probability density $\varphi(X)$ then the expected value or the mean is defined as $\mathbb{E}(X) = \int_{-\infty}^{+\infty} x\varphi(x)dx$. This is a number that is known once we know $\varphi(X)$ The variance is defined as $\mathbb{E}(X) = \int_{-\infty}^{+\infty} (x - \mathbb{E}(X))^2 \varphi(x)dx$. It is also a known number once $\varphi(X)$ is known.

A very ofthen used random continuous variable is the normal variable. It is characterised by a $\varphi(X)$ that has two parameters μ and σ, therefore we will sometimes write $\varphi_{\mu,\sigma}$. The formula for the density of the normal variable with parameters μ and σ is (don't learn this by heart)

$$\varphi_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

This normal variable has random outcomes between $-\infty$ and $+\infty$ and the mean is the parameter μ, the standard deviation is the parameter σ.

The probability that a random outcome is in the intercval $[a, b]$ is $\int_a^b \varphi_{\mu,\sigma}(x)dx$.

There are some important properties:

Property 28. *If X is a normal random variable with mean μ and standard deviation σ we write it as $X \sim N(\mu, \sigma)$* [13]

- *if we add a fixed number a to a normal random variable X then the variable $X + a$ also has random outcomes. It can be shown that it is also normal with mean $\mu + a$ and the standard deviation stays the same:*

  $X \sim N(\mu, \sigma) \implies X{+a} \sim N(\mu{+a}, \sigma)$, *no change to σ !*

- *If we multiply a random normal variable with a fixed number a then the new variable will also have random outcomes. This new random variable will also be normal with a mean and standard deviation both multiplied by a:*

  $X \sim N(\mu, \sigma) \implies X{\cdot a} \sim N(\mu{\cdot a}, \sigma{\cdot a})$

- *from this it follows that, if $X \sim N(\mu, \sigma)$ then when we define $z = \frac{X-\mu}{\sigma}$ we find that $z \sim N(0, 1)$. A random variable that is normal with a mean*

[13] In some books the second parameter is the variance, then the notation is $X \sim N(\mu, \sigma^2)$.

equal to zero and a standard deviation of 1 is called standard normal*. For a standard normal variable we use φ for the density, so we do not indicate the subscripts '0,1'.*

- *The inverse is also true: if $z \sim N(0,1)$ then $X = \mu + \sigma z$ will be Normal with mean μ and standard deviation σ.*
- *If $X \sim N(\mu, \sigma)$ then the probability that X has a value smaller than or equal to a is $P(X \leq a) = \int_{-\infty}^{a} \varphi_{\mu,\sigma}(x)dx$.*

  The probability that a normal random variable is smaller than (or equal to) a is called the cumulative probability *and is denote with ϕ, so $\phi_{\mu,\sigma}(a) = P(X \leq a)$.*

  If we do this for all possible values a then we get a function, this is the cumulative density function *of the random normal variable X.*
- *If $X \sim N(\mu, \sigma)$ then the probability that X has a value greater than a is $P(X > a) = \int_{a}^{+\infty} \varphi_{\mu,\sigma}(x)dx = 1 - \int_{-\infty}^{a} \varphi_{\mu,\sigma}(x)dx$.*
- *If $X \sim N(\mu, \sigma)$ then the probability that X has a value equal to a is $P(X = a) = \int_{a}^{a} \varphi_{\mu,\sigma}(x)dx = 0$.*

  Therefore $P(X \geq a) = P(X > a$ or $X = a)) = P(X > a) + P(X = a) = P(X > a)$.
- *If $X \sim N(\mu, \sigma)$ then the probability that X has a between a and b is $\int_{a}^{b} \varphi_{\mu,\sigma}(x)dx$.*

  This is the surface of the area under the curve of $\varphi_{\mu,\sigma}(x)$ between a and b.

We will use these properties very often, e.g. if $\mathbf{dz} \sim N(0, \sqrt{dt})$ then $\epsilon = \frac{\mathbf{dz}}{\sqrt{dt}}$ will be normal $N(0,1)$ or $\mathbf{dz} = \epsilon \cdot \sqrt{dt}$ where $\epsilon \sim N(0,1)$.

Another example, if we know that $ln(S_T) - ln(S_0) \sim N(\mu, \sigma)$ then we know that $ln(S_T) \sim N(\mu + ln(S_0), \sigma)$

6.6.3 A model for stock prices.

If we want to know how the value of a derivative evolves in the future, then, as the value of a derivative depends on the value of the underlying stock, we will have to assume some pattern for the stock price. We saw in the introduction that we will use stochastic processes to do that:

$$\overbrace{S_0}^{\text{known value}} \rightarrow \underbrace{\overbrace{S_{t_1}}^{\text{distribution}} \rightarrow \overbrace{S_{t_2}}^{\text{distribution}} \ldots \rightarrow \overbrace{S_T}^{\text{distribution}}}_{\text{Stochastic process}}$$

The point now is to find what that distribution, or better the whole series of distributions, looks like.

As a parenthesis: Think about your course on econometrics, more in particular, linear regression. There you had a dependent variable y and if you want to "estimate" how it evolves with e.g. the independent variable time, then you would make a "model" like $y = \beta_0 + \beta_1 t + u$, where $u \sim N(0, \sigma)$. But taking into account the properties of a normal distribution in the previous section this can also we written as $y \sim N(\beta_0 + \beta_1 t, \sigma)$. So for each t you have a distribution ! So in essence this is a special case of such a stochastic process.

With linear regresstion you sometimes transformed the variables, like $y' = ln(y)$ or other.

Also with linear regression you have to find a "function" that best reflects the real link between y and t.

All the reasoning below is just an application of that !

The choice of the functional form and of the transformations of the variables is of course crucial for the quality of the predictions in the future. By looking at the data, experts like Merton, Black, Scholes and others have "found" that the percentage growth[14] of the stock price between to time points is proportional to the time passed between the two time points, and to model uncertainty, there is also an error that is normal with mean 0 and a standard deviation that is proportional to the square root of the time passed: $\frac{S_t - S_0}{S_0} = \mu(t-0) + u$ where $u \sim N(0, \sigma\sqrt{(t-0)})$ but only if t is very small, so $\frac{dS}{S} = \mu df + u$ where $u \sim N(0, \sigma\sqrt{dt})$, and using the properties of a normal variable we find $\frac{dS}{S} = \mu dt + \sigma \cdot e$ where $e \sim N(0, \sqrt{dt})$.

Just to simplify the notation we will use **dz** in stead of e, but remember that **dz** is not a differential but a random variable with a standard deviation $\sqrt{dt}$ and mean zero. Therefore we will use boldface in the notation.

$$\frac{dS}{S} = \mu dt + \sigma \mathbf{dz} \text{ where } \mathbf{dz} \sim N(0, \sqrt{dt})$$

Note that there is a component that makes the percentage change in the stock price $\frac{dS}{S}$ increase in time (μ), this is called the drift (sometimes expected return). But at each pount in time we add a random components that has a normal distribution with a standard deviation that is $\sigma\sqrt{dt}$, so the stndard deviation gets larger with time (this can be seen on the figures of the stochastic processes ! σ is called the volatility.)

Using the properties of the normal distribution we can also write $\mathbf{dz} = \epsilon\sqrt{dt}$ where $\epsilon \sim N(0, 1)$, so the above formula can also be written as:

$$\frac{dS}{S} = \mu dt + \sigma\epsilon\sqrt{dt} \text{ where } \epsilon \sim N(0, 1)$$

In terms of (small) differences we will write:

[14] Note that if you want to know the percentage growth from 150 to 160 you compute (160-150)/150.

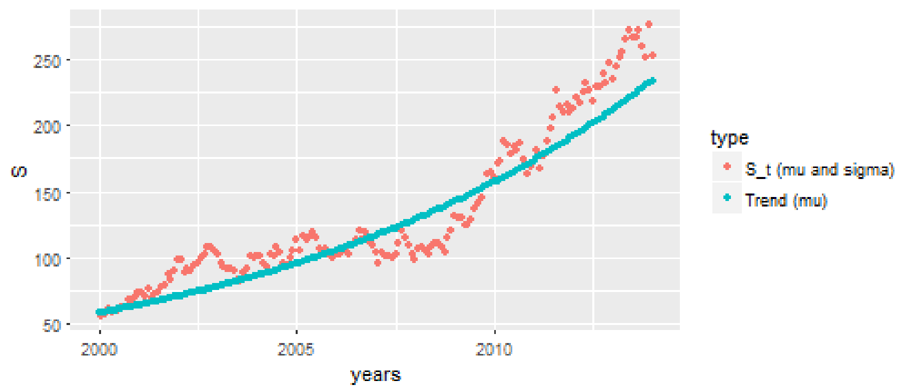
$$\frac{\Delta S}{S} = \mu \Delta t + \sigma \mathbf{\Delta z} \text{ where } \mathbf{\Delta z} \sim N(0, \sqrt{\Delta t})$$

It is very important to understand what this formula means; It says that the percentage change in the stock price in a time Δt is proportional to the length of the time interval Δt (i.e. $\mu\Delta t$) plus a deviation that is proportional to a random number drawn from a normal distribution and that also depends on Δt. Note that a random draw from any normal distribution Δa could be any number between $-\infty$ and $+\infty$, so this could be a large number !

Remark 6.5. *An important remark must be made here:*

If (!!!!) the model would have been $\frac{dS}{S} = \mu dt$ then, knowing that $d(ln(x))/dx = 1/x$, so $d(ln(x)) = dx/x$ you can see that $d(ln(S)) = \frac{dS}{S} = \mu dt$ and also $\int_0^t d(ln(S)) = \int_0^t dt$ or $ln(S_t) = ln(S_0) + \mu(t-0)$ or $S_t = S_0 e^{\mu t}$.

This is the smooth line in the graph below.



However, the model is not (!!) $\frac{dS}{S} = \mu dt$ but $\frac{dS}{S} = \mu dt + \sigma\epsilon\sqrt{dt}$ and we do not know how we can integrate $\sqrt{dt}$, nor do we know how we can integrate ϵ ???

To find a solution for that we will need Ito's lemma, therefore we give a short summary on stochastic processes.

Note, the zig-zag line in the figure is one random outcome of $\frac{dS}{S} = \mu dt + \sigma\epsilon\sqrt{dt}$, because ϵ is random there are other possible outcomes !!!

6.6.4 Stochastic processes.

We have seen that stochastic processes are a series of random variables. They can be described by their changes in time, like e.g. the dS as a function of dt supra.

We will define several kinds of stochastic process, from simple ones to more complex ones:

Definition 21. *We define the following stochastic processes:*

- *The simplest process is a standard random variable multiplied by the square root of dt. This is called a Wiener process, so a Wiener process* $\mathbf{dz}$ *is* $\mathbf{dz} = \epsilon\sqrt{dt}$ *where* $\epsilon \sim N(0,1)$.

- *A Generalised Wiener process, x(t) is a process described by* $dx = a \cdot dt + b \cdot \mathbf{dz}$*, where* $\mathbf{dz}$ *is a Wiener process and* a *and* b *are constants.*

  a is called the drift and b is called the volatility of the process. The drift describes a kind of "average" (like the smooth line in the figure supra) and b described the width of the movements around that smooth line.

  Note that in our case we have $\frac{dS}{S} = \mu dt + \sigma\mathbf{dz}$ *or* $dS = \mu \cdot S \cdot dt + \sigma \cdot S \cdot \mathbf{dz}$*, so* $a = \mu \cdot S$ *and* $b = \sigma \cdot S$ *are not constant because they depend on* S*, so we further generalise:*

- *An Ito process, x(t) is a process described by* $dx = a(x,t) \cdot dt + b(x,t) \cdot \mathbf{dz}$*, where* $\mathbf{dz}$ *is a Wiener process. So an Ito process has a non-constant drift/volatility.*

- *A Geometric Brownian Motion is a spcial kind of Ito process, namly an Ito process with drift of the form* $a(x,t) = \mu x$ *and volatility of the form* $b(x,t) = \sigma x$.

  Our stock price model in the previous section was $\frac{dS}{S} = \mu dt + \sigma\mathbf{dz}$ *or* $dS = \mu \cdot S \cdot dt + \sigma \cdot S \cdot \mathbf{dz}$*, so it is a Geometric Brownian Motion !*

We already mentioned that the prices of a derivative are a function of S and Ito's lemma tells us how a function of an Ito process changes in time:

Property 29. *This is Ito's lemma:*
If $x(t)$ is an Ito process, i.e. if $dx = a(x,t)dt + b(x,t)\mathbf{dz}$ *and G is a function of x then the change of G in time is governed by:*

$$dG = \left(a\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + b^2\frac{1}{2}\frac{\partial^2 G}{\partial x^2}\right)dt + b\frac{\partial G}{\partial x}\mathbf{dz} \text{ where } \mathbf{dz} \sim N(0,\sqrt{dt})$$

Let's give a "heuristic proof", i.e. just the big ideas;

As we will use this for x=S, we use S in stead of x.

If G is a function that is differentiable with respect to S and t, then via a Taylor expansion we find
$\Delta G = \frac{\partial G}{\partial S}\Delta S + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \frac{\partial^2 G}{\partial S \partial t}\Delta S \Delta t + \ldots$
As we let $\Delta t \to 0$ then Δt^2 goes even faster to zero, just as $\Delta t \Delta S$ and all the further terms. So we find that
$\Delta G \approx \frac{\partial G}{\partial S}\Delta S + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\Delta S^2$
If ΔS is an Ito process, then by the above definition we know that $\Delta S = a(S,t)\Delta t + b(S,t)\Delta z$ where Δz is not a differential but a

Normal random variable with a standard deviation that depends on $\sqrt{\Delta t}$.

$$\begin{aligned}\Delta G &\approx \frac{\partial G}{\partial S}(a\Delta t + b\Delta z) + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}(a\Delta t + b\Delta z)^2 \\ &\approx \frac{\partial G}{\partial S}(a\Delta t + b\Delta z) + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}(a^2\Delta t^2 + b^2\Delta z^2 + 2ab\Delta z\Delta t)\end{aligned}$$

Note that Δz is $N(0, \sqrt{\Delta t})$ so Δz can be re-written as $\Delta z = \epsilon \cdot \sqrt{\Delta t}$ where $\epsilon \sim N(0, 1)$.

$$\Delta G \approx \frac{\partial G}{\partial S}(a\Delta t + b\Delta z) + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}(a^2\Delta t^2 + b^2\epsilon^2\Delta t + 2ab\epsilon\sqrt{\Delta t}\Delta t)$$

When $\Delta t \to 0$ then powers of Δt go to zero, $\Delta t^2 \to 0$, $\Delta t^{3/2} \to 0$, so $\Delta S^2 \approx b^2\epsilon^2\Delta t$.

$$\Delta G \approx \frac{\partial G}{\partial S}(a\Delta t + b\Delta z) + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial S^2} \overbrace{b^2\epsilon^2\Delta t}^{\to b^2 dt \text{ when } \Delta t \to 0}$$

Note that $\epsilon \sim N(0, 1)$ and as $\mathbb{E}(\epsilon^2) = 1$ we find that $\mathbb{E}(\epsilon^2\Delta t) \to dt$, moreover, the variance $Var(\epsilon^2\Delta t) \to 0$ so in the limit $\epsilon^2\Delta t$ goes to a random variable with mean dt and variance zero, so it becomes dt. If you look at the normal distribution (the Gauss curve) with a certain mean and a variance (i.e. the "width" of the bell-shaped curve) of zero then this is a single value, equal to the mean, in this case that mean is dt.

$dG \approx \frac{\partial G}{\partial S}dS + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}b^2 dt = \frac{\partial G}{\partial S}(adt + b\mathbf{dz}) + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}b^2 dt$

Rearranging the terms gives the formula in the Ito Lemma.

6.6.5 Important applications of Ito's Lemma

Evolution of forward prices: We have seen that the price if a forward depends on the stock price in the following way $F = Se^{r(T-t)}$, so obviously F depends on S, so now we want to find out how F changes in time, knowing the evolution of S in time.

Assuming that the stock prices for the future follow a a Geometric Brownian Motion, i.e. $dS = \mu S dt + \sigma S \mathbf{dz}, \mathbf{dz} \sim N(0, \sqrt{df})$ we want to find a formula for dF.

It seems that is a function of S, namely $F(S) = Se^{r(T-t)}$ and because S follows (by assumption) an Ito process (because the Geometric Brownian Motion is a special kind of Ito process) we can apply Ito's lemma to $G(S) = F(S) = Se^{r(T-t)}$:

Writing $dS = \mu S dt + \sigma S \mathbf{dz}, \mathbf{dz} \sim N(0, \sqrt{df})$ in the form of an Ito process $dx = a(S,t) \cdot dt + b(S,t) \cdot \mathbf{dz}$ it follows that $a(S,t) = \mu S, b(S,t) = \sigma S$.
For a forward we can see that $G(S) = Se^{r(T-t)}$.
Ito's lemma tells us that (because S is an Ito process) the price of the forward changes in time as :
$dG = \left(a\frac{\partial G}{\partial S} + \frac{\partial G}{\partial t} + b^2 \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \right) dt + b\frac{\partial G}{\partial S}\mathbf{dz}$
We know that (cfr supra) $a(S,t) = \mu S, b(S,t) = \sigma S$, so the only thing we need are the partial derivatives of G where $G(S) = Se^{r(T-t)}$.
Using the rules for derivation we find $\frac{\partial G}{\partial S} = e^{r(T-t)}$, $\frac{\partial G}{\partial t} = -rF$, $\frac{\partial^2 G}{\partial S^2} = 0$.
Substituting all this into the formula above we find:
$dG = \left(\mu \overbrace{Se^{r(T-t)}}^{F} - rF \right) dt + \sigma \overbrace{Se^{r(T-t)}}^{F} \mathbf{dz} = (\mu - r)F dt + \sigma F \mathbf{dz}$

Since we defined (see supra) $G(S) = F(S)$ we find that $dG = dF$ so

$$\frac{dF}{F} = (\mu - r)dt + \sigma \mathbf{dz}$$

At first glance this looks just like a formula, but this is a very important result:

Property 30. *We find:*

We started from a model of our stock prices, it was a stochastic process (a Geometric Brownian Motion) that described how the stock price in the future may look like and the uncertainty we have about these prices in the future, and now , using that and Ito's lemma, we find how the forward prices will evolve in the future , it is also a Geometric Brownian Motion with drift $\mu - r$ and with the same volatility σ.

$$\overbrace{S_0}^{\text{known value}} \rightarrow \underbrace{\overbrace{S_{t_1}}^{\text{distribution}} \rightarrow \overbrace{S_{t_2}}^{\text{distribution}} \ldots \rightarrow \overbrace{S_T}^{\text{distribution}}}_{\text{Stochastic process}}$$

$$\Longrightarrow \overbrace{F_0}^{\text{known value}} \rightarrow \underbrace{\overbrace{F_{t_1}}^{\text{distribution}} \rightarrow \overbrace{F_{t_2}}^{\text{distribution}} \ldots \rightarrow \overbrace{F_T}^{\text{distribution}}}_{\text{Stochastic process}}$$

And when we know the distribution of the forward prices, we can compute the probability that this price in the future will be more than ...

Continuous returns: But there is more ...

However, we already stated that, if the spotprice of a stock is Geometric Brownian Motion, so $\frac{dS}{S} = \mu dt + \sigma\epsilon\sqrt{dt}$, but we do not know how we can integrate $\sqrt{dt}$, nor do we know how we can integrate ϵ !!

Ito's lemma will allow us to do so:

Let us assume that S follows a Geometric Brownian Motion, i.e. $S = \mu S dt + \sigma S \mathbf{dz}, \mathbf{dz} \sim N(0, \sqrt{dt})$. And let us look at the evolution of $G = ln(S)$.
As in the previous example we can see that this is an Ito process with $a(S,t) = \mu S, b(S,t) = \sigma S$.
As $G(S) = ln(S)$ it holds that $\frac{\partial G}{\partial S} = \frac{1}{S}$, $\frac{\partial G}{\partial t} = 0$ and $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$
Ito's lemma says (because S is an Ito process):
$dG = \left(\mu S \frac{1}{S} - \frac{1}{2}(\sigma S)^2 \frac{1}{S^2}\right) dt + \sigma S \frac{1}{S}\mathbf{dz}$
or we find that

$$dG = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma \mathbf{dz}$$

As $\mathbf{dz} \sim N(0, \sqrt{dt})$ it follows that:
$dG \sim N(\left(\mu - \frac{1}{2}\sigma^2\right) dt; \sigma\sqrt{dt})$ or in terms of (small) differences:

$$\Delta G \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t; \sigma\sqrt{\Delta t}\right)$$

Since have choosen $G(S) = ln(S)$ and let S_0 be the stock at $t = 0$ and S_T the stock at $t = T$, then $\Delta G = G(S_T) - G(S_0) = ln(S_T) - ln(S_0)$ and $\Delta t = T - 0$, so we have $ln(S_T) - ln(S_0) \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right) T; \sigma\sqrt{T}\right)$ or $ln(S_T) \sim N\left(ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right) T; \sigma\sqrt{T}\right)$

Property 31. *If the stock price follows a Geometric Brownian Motion, i.e.* $dS = \mu S dt + \sigma S\mathbf{dz}, \mathbf{dz} \sim N(0, \sqrt{dt})$*, then*

$$d(ln(S)) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\mathbf{dz}, \mathbf{dz} \sim N(0, \sqrt{dt})$$

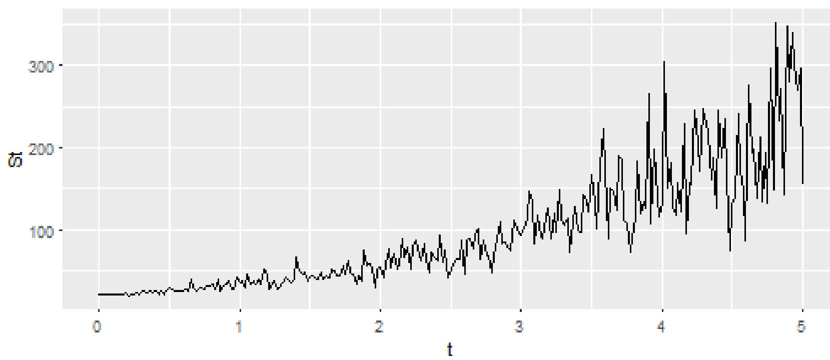
and also

$$ln(S_T) \sim N\left(ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right) T; \sigma\sqrt{T}\right)$$

Note that this is no longer about dS, but about S_T for any future time T !! So we have found a way to "integrate" the Geometric Brownian motion !!
Note that S_T is not one value but a distribution , so an infinite number of values, each with its own probability!!
This result and the tables of the normal distribution can be used to find the probabilities that $ln(S_T)$ is in a certain interval. μ and σ must be estimated form historical data on stock prices, T is known and S_0 is also known.

Definition 22. *If $ln(S_T)$ has a normal distribution with parameters μ and parameters σ then we say that S_T has a lognormal distribution.*

Using the property of the normal distribution we can write:
$ln(S_T) \sim ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\epsilon$ where $\epsilon \sim N(0,1)$. With this we can do monte-carlo simulations, e.g. assume we use historical data and find that $\mu = 12\%$ and $\sigma = 15\%$ and let $S_0 = 20$, then we find one possible path:



You can clearly see the drift (average upward trend) and the volatility (spread), the latter increases with t (because of the $\sigma\sqrt{t}$, so the standard deviations of each distribtion of S_t becomes wider as time moves on).

An example of how we can use this: (Hull 11.16)
Assume that the stock price is log-normal, i.e.
$ln(S_T) \sim N\left(ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)T; \sigma\sqrt{T}\right)$.
The current stock price is 50, return is 12% and volatility is 30%. What is the probaility that the stock price will be above 80 in two years ?

So we have $S_0 = 50, \mu = 0.12, \sigma = 0.3, T = 2$

$$ln(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)T = ln(50) + \left(0.12 - \frac{1}{2}0.3^2\right)2 = 4.062023$$

$$\sigma\sqrt{T} = 0.3\sqrt{2} = 0.4242641$$

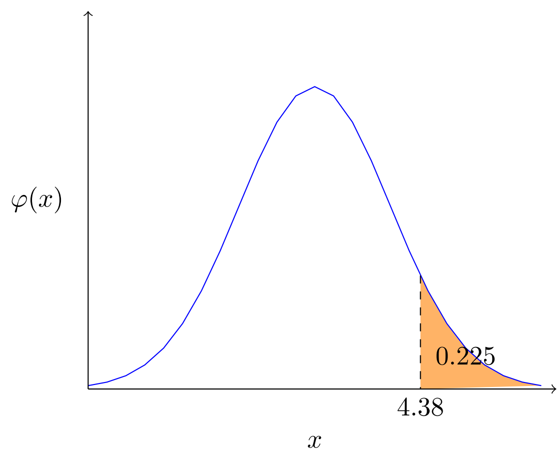
So by the above we know that $ln(S_T) \sim N(4.062023, 0.4242641)$

$$S_T \geq 80 \iff ln(S_T) \geq ln(80) = 4.3820266$$

We know the probabilities of $ln(S_T)$ because we know its distribution namely Normal with mean 4.062023 and standard deviation 0.4242641. So we can compute the probabilty that

$$P(S_T \geq 80) = P(ln(S_T) \geq 4.3820266) = \int_{4.3820266}^{+\infty} \varphi(x)dx = 0.2253479.$$

This integral can be found using the tables of the normal distribution and/or with a calculator.



You can also compute the 95% interval.

A note on $d(ln(S))$, with S a stochastic process: If S would be a function then $d(ln(S))$ is dS/S but from the above we see that this does not hold when S is a stochastic process. You can not apply the rules of "normal derivation" to stochastic processes, you must use Ito's lemma !

Indeed, the Geometric Brownian Motion for the stock price S with drift μ and volatility σ is $\frac{dS}{S} = \mu dt + \sigma \mathbf{dz}$.

With Ito's lemma we found $d(ln(S)) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma \mathbf{dz}$

if $d(ln(S))$ would be equal to dS/S as would be the case if the normal rules of derivation would hold, then both can not be true because we would then find two different expressions of dS/S.

So be carefull to apply rules of normal derivation when working with stochastic processes !!

Why is this ?

- Well if S is not a stochastic process then you can apply the normal rules, so $d(ln(S)) = dS/S$ so if this would be equal to μdt , without the random term in dz then after integration you would find $ln(S_T) = ln(S_0) + \mu T$ or $S_T = S_0 e^{\mu T}$ when no stochatsic process !! Because there is no stochastic term the same μ holds everywhere between 0 and T. e.g. if $\mu = 3\%$ and $T = 2$ and $S_0 = 1$ then $S_T = e^{0.03 \times 2} = 1.0618365$.
- if S is a stochastic process with $dS/S = \mu dt + \sigma \mathbf{dz}$ then we are adding an random value $\sigma \mathbf{dz}$ to the μ and that random value os zero on average and symmetric (normal variable is symmetric). So it is a μ that is "on average" equal to 3% and e.g. in the first year it could be 2.5% and in the second 3.5% (this is on average 3%)

The growth between 0 and $T = 2$ would now be $e^{0.025\times1}e^{0.035\times1} = 1.059715$.

So with the stochastic process it is lower !!! (this is where the $\mu - \frac{1}{2}\sigma^2$ comes from)

> But be carefull with derivations and integrations when working with stochastic processes , you can recognise the stochastic processes when you have a $\mathbf{dz}$ or a $\sqrt{dt}$!!

The evolution of forward prices: Note that supra we found if the stock prices follow a Geometric Brownian Motion with drift μ and σ, i.e. $\frac{dS}{S} = \mu dt + \sigma\mathbf{dz}$ then the forward prices for futuire periods are governed by a stochastic process defined as $\frac{dF}{F} = (\mu - r)dt + \sigma\mathbf{dz}$, this is ... a Geometric Brownian Motion with drift $\mu' = \mu - r$ and volatility σ.

Therefore, if we apply Ito's lemma to $G(F) = ln(F)$ we will find in a similar way an expression for the foward prices:

Property 32. *If the spot prices follow a Geometric Brownian Motion with drift μ and volatility σ, then the forward prices in future time periode T are described by the following distribution:*

$$ln(F_T) \sim N\left(ln(F_0) + \left(\mu - r - \frac{1}{2}\sigma^2\right)T; \sigma\sqrt{T}\right)$$

So we can compute probabilities that the forward price is lower than ....

6.6.6 The Black-Scholes model.

It is good to read section 6.6.1 first.

We have seen that:

- The stock price is usually modelled as a Geometric Brownian Motion with drift μ and volatility σ, these parameters can be estimated from historical data, i.e. $\frac{dS}{S} = \mu dt + \sigma\mathbf{dz}$;
- In the previous chapters we also have seen that prices of derivatives can be found as a function of the underlying, so if the underlying is a share of a stock, then the price of the derivative is a function of S, let's say $G(S)$,
- So the price of a derivative is a function $G(S)$ of S and with Ito's lemma we found that (note that $a(S,t) = \mu S, b(S,t) = \sigma S$ and x is S)

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S\mathbf{dz}$$

This equation describibes the evolution of the price of a derivative.

We know that in the above formulas, the "uncertainty" is in the parts with $\sigma \mathbf{dz}$, this is why we put them in red color.

Moreover, we already have talked about a "risk neutral world" where probabilities are in "other units", and we know that, when working with stochastic processes, the probabilities are in the term $\sigma \mathbf{dz} = \sigma \cdot \epsilon \sqrt{dt}$, where $\epsilon \sim N(0,1)$.

Remember form the Binomial option model that we "entered" this risk free world by finding "a replicating portfolio" that excluded risk. We also saw that this replicating portfolio was built up of Δ (a number here) shares of the stock, combined with a European call (and this we did because that was the way we found a price of a forward).

We will now do exactly the same, only this Δ will be a bit more complicated because we have to re-compute and re-compose it in every dt interval. So it will be changing "instantauously". How do we do this ?

Well, above we found that the option price is described as:

$$\frac{dS}{S} = \mu dt + \sigma \mathbf{dz}$$

while the price of any derivative is a function $G(S)$ that evolves as:

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S \mathbf{dz}$$

We want to go to a "risk neutral" situation so we would like to get rid of the uncertainty. So if we multiply the first equation by $\frac{\partial G}{\partial S}$ and then subtract if from the second one, then the $\mathbf{dz}$ will disappear in the result !! So there is no $\mathbf{dz}$ or no uncertainty anymore.

Multilpying the first equation by $\frac{\partial G}{\partial S}$ means that we compute for $\frac{\partial G}{\partial S}$ shares of stock and if we buy that then we have to borrow the money $\frac{\partial G}{\partial S} S$ to buy the shares. At the same time we short a derivative that matures at dt.So the value of our portfolio is $\Pi = -G(S) + \frac{\partial G}{\partial S} S$.

The change in value of the portfolio is $d\Pi = dG - \frac{\partial G}{\partial S} dS$. If we substitute the expressions of dG and dS that we have found supra then this is:

$$\begin{aligned}
d\Pi &= -dG + \frac{\partial G}{\partial S} dS \\
&= -\left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \cancel{\frac{\partial G}{\partial S} \sigma S \mathbf{dz}} + \frac{\partial G}{\partial S} (\mu S dt + \cancel{\sigma S \mathbf{dz}}) \\
&= -\left(\cancel{\frac{\partial G}{\partial S} \mu S} + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \cancel{\frac{\partial G}{\partial S} \mu S} dt \\
&= -\left(\frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt
\end{aligned}$$

As expected there is no uncertainty ($\mathbf{dz}$) anymore, so by hedging we found a portfolio that no longer contains uncertainty and because of the arbitrage argument (as for the Binomial model) this shoud have a return equal to the risk free interest rate, i.e. $d\Pi$ should be $r\Pi dt$:

So we find

$$\begin{aligned} d\Pi &= -\left(\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right) dt \\ &= r\Pi dt \\ &= r(-G + \frac{\partial G}{\partial S} S)dt \end{aligned}$$

It follows that:

Property 33. *If the price of the underlying stock follows a Geometric Brownian Motion with drift μ and volatility σ i.e. $\frac{dS}{S} = \mu dt + \sigma \mathbf{dz}$ then the price of a derivative $G(s)$ must be a solution to the differential equation:*

$$\frac{\partial G}{\partial S} rS + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 = rG$$

This is the Black-Scholes differential equation, all derivatives have a price $G(S)$ that must be a solution of that equation. The underlying uncertainty is not in the equation because we contructed a portfolio such that the amout of stock and the derivative have the same uncertainty, so the underlying source of uncertainty is the same, it is eliminated by constantly re-composing the portfolio.

Note that the risk free rate was assumed to be constant !

Remark 6.6. *Several important remarks must be made here:*

- *Note that the drift of the stock price is μ, but the (Parabolic - because of order 2) Differential Equation (PDE) does not contain μ ! It only contains r, because of the risk neutrality !*

  There is no random component dz neither !!

- *note that the number of shares Δ in the Binomial trees has no become $\partial G/\partial S$ with changes every dt because $S(t)$! (note this, later we will talk about delta of portfolio)*

- *This is because we "eliminated" the uncertainty* $\mathbf{dz}$*;*

- *It is a second order differential equation, so we will need two "side conditions" to solve it, one is the value of the derivative today, the other one is the value at the maturity.*

[[this is explained below, so don't worry too much:]]
Feynman-Kac showed how to solve a differential equation of the form:

$$\frac{\partial f}{\partial t} + a(S,t)\frac{\partial f}{\partial S} + \frac{1}{2}b^2(S,t)\frac{\partial^2 f}{\partial S^2} - rf = 0$$

Subject to a final (i.e. at T) condition that $f(S, t = T) = g(S)$ is given by:

$$f(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left(g(S_T|_{S_t=S})\right)$$

Where $S(t)$ is the solution of the process

$$dS^{\mathbb{Q}} = a(S,t)dt + b(S,t)\mathbf{dz}^{\mathbb{Q}}$$

The superscript $\mathbb{Q}$ means that we have changed the measure of the probabilities !

We will explain what Feynman-Kac says, using our own "problem", i.e. the Black-Scholes differential equation:

- We can see that the Black-Scholes differential equation is of the same shape as the one mentioned by Feynman-Kac:
  - $G(S,t)$ plays the role of $f(S,t)$, $G(S,t)$ is the price of a derivative (a forward or an option) and we know that it is dependent on the price of the underlying stock (whose price is modelled as $dS/S = \mu dt + \sigma dz$) and on the timepoint where we compute it;
  - rS plays the role of $a(S,T)$, σS the one of $b(S,t)$;
  - Feynman-Kac say that we need to know $f(S,t)$ at $t = T$ as a function of S (which is S_T at $t = T$), and we call that $g(S)$, this no longer depends on T because we set $t = T$. Translating this to our setting, we need to know $G(S,t)$ at $t = T$ and call that $g(S)$. Now, $G(S,t)$ is the price of a derivative and we have to know it at $t = T$. Let's take a call option as an example; we know that at $t = T$ the price is $max(S_T - K, 0)$ and this is $g(S)$, so $g(S) = max(S - K, 0)$;

    In other words, $g(S)|_{S_0}$ is the distribution of the price of the call option at T, given that the spot price today is S_0 , this is illustrated in the graphs further on, the price paths start at S_0 and end up at T in different values because of the random component dz. This gives the distribution of S_T at T and with that you get the distribution of the call option prices at T.
  - and then Feynman-Kac says that $G(S,t)$, the price of the call option at t can be found as $G(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(g(S_T)|_{S_t=S})$, where $S(t)$ is the solution of $dS^{\mathbb{Q}} = rSdt + \sigma Sdz^{\mathbb{Q}}$. This is complicated but after explanation it will become clear. Note first that the superscript $\mathbb{Q}$ is used to make a distinction between the equation that Feynman-Kac talk about and the stochastic model of our stock price $dS = \mu Sdt + \sigma dz$.

Nevertheless, it is the same S, so $dS^{\mathbb{Q}} = dS$ but ... measured with other, risk neutral, probabilities. That is why μ has become r: in a risk neutral world there is no "premium" on top of the riskless rate r. Note that this is on fact the same world, only that we measure probabilities in different units !

But, applying Ito's lemma to $d(ln(S))$ we found (see property 31) that, if $dS = rSdt+\sigma Sdz^{\mathbb{Q}}$, then $ln(S_T) \sim N(ln(S_0)+(r-\frac{\sigma^2}{2})T; \sigma\sqrt{T})$ in other words, $ln(S_T) = ln(S_0) + (r - \frac{\sigma^2}{2})T + \sigma\sqrt{T}d\epsilon$ or

$$S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}d\epsilon}$$

Feynman-Kac says that $G(S,t)$, the price of the call option can be found as $G(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(g(S_T)|_{S_t=S})$, where $S(t)$ is the solution of $dS^{\mathbb{Q}} = rSdt + \sigma Sdz^{\mathbb{Q}}$ and we find that the solution of $dS^{\mathbb{Q}} = rSdt + \sigma Sdz^{\mathbb{Q}}$ is $S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}d\epsilon}$

So using Feynman-Kac we find that the price of the derivative that we are looking for $G(S,t)$ can be found as : $G(S,t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(g(S_T)|_{S_t=S})$

We have already found above that $g(S) = max(S-K,0)$ and with the above expression for the stock price S we have $g(S_T)|_{S_0} = max(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}d\epsilon} - K, 0)$

Therefore we can find the price of the call option[15] as (the '|' reads as "given that"):

$$\underbrace{G(S_0,t)}_{\substack{\text{value of the call} \\ \text{(with maturity at T) today}}} = \underbrace{\underbrace{\mathbb{E}^{\mathbb{Q}}(\underbrace{g(S_T)|_{S_t=S_0}}_{\substack{\text{distribution of option price at T,} \\ \text{given that stock price todat is } S_0}})}_{\text{Expected (risk neutral) value of option price at T ...}} \quad e^{r(T-0)}}_{\text{discounted value of expected value of ...}}$$

where $g(S_T) = max(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}d\epsilon} - K, 0)$ is the distribution of the price of the call at T.

This was all technical stuff, but let us now look at what this means, because it looks very complex but it is very similar to what we had with the Binomial trees !!!!

$\mathbb{E}^{\mathbb{Q}}(g(S_T)|_{S_t=S_0})$ is an expected value, so for a continuous distribution this is an integral (see statistics summary). If you look at $g(S_T)$ above, you can see that is it is the price of the option at $t = T$, but there is one strange element, $d\epsilon$ which we know is NOT a differential but a standard normal distribution !!!

[15] This also works for a forward, but with a simpler formula.

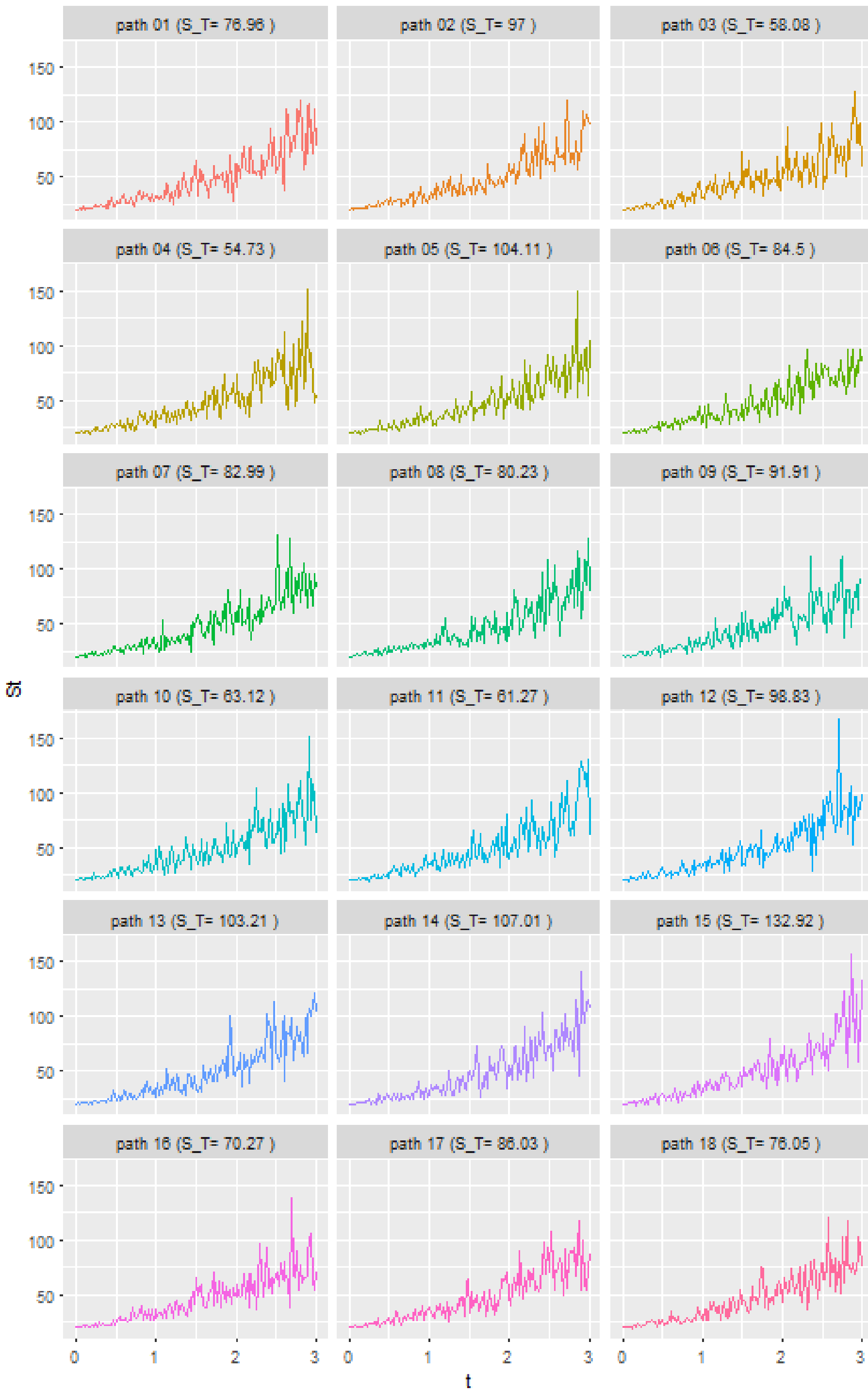
So $g(S_T)$ is the distribution of all the possible values for S_T at $t = T$.
This is similar to the Binomial tree: we had all the values of S_T at $t = T$ and for each value we had the probability. After that we took the expected value as $\sum_i p_i x_i$ where the probabilities p_i are measured in the risk neutral world !
So $\mathbb{E}^{\mathbb{Q}}$ is just taking the expected value of all the possible values for S_T at $t = T$ in the future, using probabilities that are measured with the risk neutral measure $\mathbb{Q}$. This mesure is defined by $dz^{\mathbb{Q}}$ which is a standard normal variable with density $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x^2}$.
So what we have to do is take the expected value of all the possible values of S_T at $t = T$ using the probabilities given by $\varphi(x)$, and this is very similar to what we did with the Binomial trees !
so we find that

$$\mathbb{E}^{\mathbb{Q}}(g(S_T)|_{S_t=S_0}) = \int_{-\infty}^{+\infty} g(S_T)\varphi(\epsilon)d\epsilon$$

where $g(S_T)|_{S_t=S_0} = max(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}d\epsilon} - K, 0)$
And then we have to discount this value.

It is good to get a feeling about what is happening. We know that the stock price evolves according to a Geometric Brownian motion as shown in the figure below. We know that each of these paths betwen $t = 0$ and $t = T$ are possible and that many more of these paths are possible.

So, between $t = 0$ where we know the value of the spot price and $t = T$ where many values are possible, there are an infinite number of paths between $[0, T]$ that lead to an infinite number of values for S at $t = T$, Note that at $t = T$ we have MULTIPLE VALUES that are possible, as can be seen in these four examples at the right side of the graphs (the values of S_T are in the title of the subgraph)
So what we are doing is, for a possible value of S at T (and there are many values possible) we just try to find the probability to end in that particluar value and this can be done by counting how many percent of all the (infinitely) possible paths end in that one value that is what the integral does for us and that is what the Binomial probabilities did in the Binomial tree model !
When we have the distribution of the spot price S_T at $t = T$ then we can find the distribution of the price of the call option at T using the function $g()$.
So this method can also be used to **compute the integral by Monte-Carlo simulation**; use a computer to generate an high number of such paths and take, for each path, the value **spot** S_T at T. This gives many values for S_T and thus the simulated distribution of the spot at T.
Each of these values of the spot can then be used to compute a value for the call and in that way we get the simulated distribution of the **call option** prices at T (this is $g(S)|_{S_t}$). The average of all these simulated values for the call price obtained at T is an approximation for $\mathbb{E}^{\mathbb{Q}}(g(S_T)|_{S_t=S_0})$. The more paths you simulated the better the approximation.


path 01 (S_T= 76.96)
path 02 (S_T= 97)
path 03 (S_T= 58.08)
path 04 (S_T= 54.73)
path 05 (S_T= 104.11)
path 06 (S_T= 84.5)
path 07 (S_T= 82.99)
path 08 (S_T= 80.23)
path 09 (S_T= 91.91)
path 10 (S_T= 63.12)
path 11 (S_T= 61.27)
path 12 (S_T= 98.83)
path 13 (S_T= 103.21)
path 14 (S_T= 107.01)
path 15 (S_T= 132.92)
path 16 (S_T= 70.27)
path 17 (S_T= 86.03)
path 18 (S_T= 76.05)
St
t
150
100
50
0
1
2
3

Write expectation as an integral: The expected value is an integral (see summary on statistics) and we know that for a European Call it holds that $G(S) = max(S_T - K, 0)$, so we find that:

$$\mathbb{E}^{\mathbb{Q}}(g(S_T)|_{S_t=S_0}) = \int_{-\infty}^{+\infty} \left(max\left(0, S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}\epsilon} - K\right)\right) \varphi(\epsilon) d\epsilon$$

Split the integration interval: Note that $max\left(0, S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}\epsilon} - K\right)$ is zero when $K \geq S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}\epsilon}$, or $e^{\sigma\sqrt{T}\epsilon} \leq K/S_0 e^{-(r-\sigma^2/2)T}$ or, after taking 'ln', $\epsilon \leq \frac{ln(K/S_0)-(r-\sigma^2/2)T}{\sigma\sqrt{T}}$

so we find: $max\left(0, S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}\epsilon} - K\right) = 0 \iff \epsilon \leq \frac{ln(K/S_0)-(r-\sigma^2/2)T}{\sigma\sqrt{T}}$

and otherwise it is $S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}\epsilon} - K$

Subsituting this in the integral above we find:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}(G(S_T)) &= \int_{-\infty}^{\epsilon_0} 0\varphi(\epsilon) d\epsilon \\ &+ \int_{\epsilon_0}^{+\infty} \left(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}\epsilon} - K\right) \varphi(\epsilon) d\epsilon \\ &= \int_{\epsilon_0}^{+\infty} \left(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}\epsilon} - K\right) \varphi(\epsilon) d\epsilon\end{aligned}$$

where $\epsilon_0 \leq \frac{ln(K/S_0)-(r-\sigma^2/2)T}{\sigma\sqrt{T}}$.

Split the remaining integral: Let us look at the integral:
$\int_{\epsilon_0}^{+\infty} \left(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}\epsilon} - K\right) \varphi(\epsilon) d\epsilon = \int_{\epsilon_0}^{+\infty} S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}\epsilon} \varphi(\epsilon) d\epsilon -$
$\int_{\epsilon_0}^{+\infty} K\varphi(\epsilon) d\epsilon = S_0 \underbrace{\int_{\epsilon_0}^{+\infty} e^{(r-\sigma^2/2)T+\sigma\sqrt{T}\epsilon} \varphi(\epsilon) d\epsilon}_{I_2} - K \underbrace{\int_{\epsilon_0}^{+\infty} \varphi(\epsilon) d\epsilon}_{I_1} = S_0 I_1 -$
KI_2

Now $I_1 = \int_{\epsilon_0}^{+\infty} \varphi(\epsilon) d\epsilon$ where $\varphi(\epsilon)$ is $P(z \geq \epsilon_0)$ where z is standard normal. Because of the symmetry of the Gauss curve we have $P(z \geq \epsilon_0) = P(z \leq -\epsilon_0)$ and this is $N(-\epsilon_0)$ where N is the cumulative density of the standard normal.

$I_2 = \int_{\epsilon_0}^{+\infty} e^{(r-\sigma^2/2)T+\sigma\sqrt{T}\epsilon} \varphi(\epsilon) d\epsilon = \underbrace{e^{(r-\sigma^2/2)T}}_{\text{no } \epsilon} \int_{\epsilon_0}^{+\infty} e^{\sigma\sqrt{T}\epsilon} \varphi(\epsilon) d\epsilon.$

We know that $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}x^2}$, therefore $\int_{\epsilon_0}^{+\infty} e^{\sigma\sqrt{T}\epsilon} \varphi(\epsilon) d\epsilon = \int_{\epsilon_0}^{+\infty} e^{\sigma\sqrt{T}\epsilon} \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\epsilon^2} d\epsilon =$
$\frac{1}{\sqrt{2\pi}} \int_{\epsilon_0}^{+\infty} e^{\sigma\sqrt{T}\epsilon + \frac{1}{2}\epsilon^2} d\epsilon$

so $I_2 = e^{(r-\sigma^2/2)T}\frac{1}{\sqrt{2\pi}}\int_{\epsilon_0}^{+\infty} e^{\sigma\sqrt{T}\epsilon+\frac{1}{2}\epsilon^2}d\epsilon$

Note that $\sigma\sqrt{T}\epsilon+\frac{1}{2}\epsilon^2 = -\frac{1}{2}(\epsilon-\sigma\sqrt{T})^2+\frac{1}{2}\sigma^2 T$. or $I_2 = e^{(r-\sigma^2/2)T}\frac{1}{\sqrt{2\pi}}\int_{\epsilon_0}^{+\infty} e^{-\frac{1}{2}(\epsilon-\sigma\sqrt{T})^2+\frac{1}{2}\sigma^2 T}d\epsilon =$
$e^{(r-\sigma^2/2)T}\underbrace{e^{\frac{1}{2}\sigma^2 T}}_{\text{no }\epsilon}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{\epsilon_0}^{+\infty} e^{-\frac{1}{2}(\epsilon-\sigma\sqrt{T})^2}d\epsilon}_{P(X\geq\epsilon_0),\text{with } X\sim N(\sigma\sqrt{T};1)}$

Further, if $X \sim N(\sigma\sqrt{T};1)$ then $X-\sigma\sqrt{T} \sim N(0,1)$ and $P(X \geq \epsilon_0) = P(z = X-\sigma\sqrt{T} \geq \epsilon_0-\sigma\sqrt{T})$, where $z \sim N(0,1)$.

So we need $P(z \geq \epsilon_0-\sigma\sqrt{T})$ and by because the normal distribution is symmetric this is the same as $P(z \leq -\epsilon_0+\sigma\sqrt{T})$ which, if N is the cumulative distributin of z is $N(-\epsilon_0+\sigma\sqrt{T})$

So $I_2 = e^{(r-\sigma^2/2)T}e^{\frac{1}{2}\sigma^2 T}N(-\epsilon_0+\sigma\sqrt{T}) = e^{rT}N(-\epsilon_0+\sigma\sqrt{T})$

Put it all together: $G(S,t) = (S_0\cdot I_2 - K\cdot I_1)e^{-r(T-0)}$,

where $I_1 = N(-\epsilon_0)$ and $I_2 = N(-\epsilon_0+\sigma\sqrt{T})e^{rT}$ and $\epsilon_0 = \frac{ln(K/S_0)-(r-\sigma^2/2)T}{\sigma\sqrt{T}}$

So $G(S,t) = (S_0\cdot N(-\epsilon_0+\sigma\sqrt{T})e^{rT} - K\cdot N(-\epsilon_0))e^{-r(T-0)} = S_0\cdot N(-\epsilon_0+\sigma\sqrt{T}) - K\cdot N(-\epsilon_0)e^{-r(T-0)}$ where $\epsilon_0 = \frac{ln(K/S_0)-(r-\sigma^2/2)T}{\sigma\sqrt{T}}$

to find the formula of Black and Scholes we have to write put $d_1 = -\epsilon_0+\sigma\sqrt{T} = -\frac{ln(K/S_0)-(r-\sigma^2/2)T}{\sigma\sqrt{T}} + \sigma\sqrt{T} = \frac{-ln(K/S_0)+(r-\sigma^2/2)T+\sigma^2 T}{\sigma\sqrt{T}} = \frac{ln(S/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$

if $d_1 = -\epsilon_0+\sigma\sqrt{T}$ then $-\epsilon_0 = -\sigma\sqrt{T}+d_1$ and we call this d_2

so we find

Remark 6.7. *Note that , looking the reasoning in the proof, the price of a forward is a special case where* $\epsilon_0 = -\infty$

Black-(Merton)-Scholes:

$$EC_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$EP_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

- $d_1 = \frac{ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$
- $d_2 = d_1 - \sigma\sqrt{T}$
- N is the cumulative density of a standard normal random variable.

Note that from the proof it follows that

- $N(d_2) = I_1$ which is the risk neutral probability that the option will be exercised.
- $S_0 N(d_1) e^{rT}$ is the expected stock price (risk neutral expectation) at T when stock prices les than the strike price K are counted as zero.

Forward as a special case ?
draw distributions of F_T, S_T, EC_T?

Example: As an example assume the the stock price today is $S_0 = 20$. The voltatility of the stock is $\sigma = 0.15$.

We want to find today's value of a call option with maturity $T = 5$ and a strike price $K = 10$.

The risk free interest rate is $r = 0$

Let's apply the formula that Black-Scholes found; $d_1 = 2.2342707, d_2 = 1.8988605, N(d_1) = 0.9872674, N(d_2) = 0.9712086$[16]. We find that $EC_0 = 10.033$.

Alternative: Simulate the answer:

We could also find the solution by simulation. For that we have to find the distriution of the value of the call option at T, i.e. $g(S_T)$, i.e. $g(S_T)|_{S_t=S_0} = max(S_T - K, 0)$ where we know that S_T (a value in the furtue) is a distribution. Indeed, we know that $S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}d\epsilon} - K)$ and $\epsilon \sim N(0,1)$ is a normal random variable.

So if we take $n = 500,000$ random values from a standard normal distribution , then we can find n values for S_T (and thus a distribution) and also n values for EC_T (thus the distribution of EC_T).

[16] In Excel the function 'NORM.DIST' can be used to compute $N(x)$: '=NORM.DIST('x',0,1,TRUE)'.

These are shown in the graph below, EC_0 are the discounted values of EC_T.

These densities are shown in the graph below, on the horizontal axis you see the different values that might come out as S_T (left panel inthe graph) or EC_T (right panel in the graph) and on the vertical axis the percentage of times (probability) that this value occurs. E.g. for EC_T the value 10 occurs around 7pct. of the time as value for EC_T at the time T (in the furture).

If we now compute the average of all these EC_T values and then discount that average, then we get the simulated value, we find 10.032. Note that it is very close to the value found with the formula of Black-Scholes !

BUT THERE IS MORE: with the simulated distribution you can see that the option can sometimes be worthless (see the density above 0 in the graph of the density of EC_T) because of the volatility in the stock price !

Moreover we have an idea about the spread of the outcome of EC_T and this about the "precision" of our value EC_0 !!!

Note that with such distributions you can compute the probability that S_T is above ... or that EC_T is above ...

7 NEW:Some extensions.

7.1 A special case: options on indexes.

Take care with β ! I don't see anything in the slides on this, did he talk about it ? (see Hull 334-335, it is easy)

7.2 A special case: options on continuous dividend paying stock.

7.2.1 Binomial model.

Very similar, the only difference is the risk neutral probability, see exercise 11.5.2.

If we have a European call option on an underlying stock that pays a continous dividend yield q i.e. a the dividend is a percentage value of the stock, then (see figure 8) the value of the portfolio after one period will be $\Delta S_0 u e^{q\Delta t} + B e^{r\Delta t}$ after an upward movement and $\Delta S_0 d e^{q\Delta t} + B e^{r\Delta t}$.

So, with a similar reasoning as after that figure we find two equations:

$$\begin{aligned} \Delta \cdot S_0 \cdot u \cdot e^{q\Delta t} + B \cdot e^{r \cdot \Delta t} &= EC_u \\ \Delta \cdot S_0 \cdot d \cdot e^{q\Delta t} + B \cdot e^{r \cdot \Delta t} &= EC_d \end{aligned}$$

So, what your professor has on his slides is top copy the reasoning in figure 8 and do the same computations but where you replace u by $u \cdot e^{q\Delta t}$ and d by $d \cdot e^{q\Delta t}$ and then do exactly the same computations. You can do these computations (see his slides).

It is however much easier to say that $u' = u \cdot e^{q\Delta t}$ and $d' = d \cdot e^{q\Delta t}$ are the upward/downward increase/decrease and then , in the result of the computations do the inverse replacement.

$$\begin{aligned} \Delta \cdot S_0 \cdot \overbrace{u \cdot e^{q\Delta t}}^{u'} + B \cdot e^{r \cdot \Delta t} &= EC_u \\ \Delta \cdot S_0 \cdot \overbrace{d \cdot e^{q\Delta t}}^{d'} + B \cdot e^{r \cdot \Delta t} &= EC_d \end{aligned}$$

We now have exactly the same equations as after figure 8) except that there is u' and d' in stead of u and d, se we have to do the "inverse subsitutions" to find again u and d in our formula.

E.g. we found that the risk neutral probability is (with u' and d') $p = \frac{e^{r\Delta t} - d'}{u' - d'}$, if we replace u' and d' then we find that $p = \frac{e^{r\Delta t} - d \cdot e^{q\Delta t}}{u \cdot e^{q\Delta t} - d \cdot e^{q\Delta t}}$ and after dividing numerator and denominator by $e^{q\Delta t}$ we find

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d}$$

, where u and d are determined as before (see section 6.4)!

So the result is exactly the same as before, execpt that in the exponent of the risk neutral probability you now have $r - q$ in stead of r.

So we find that the risk neutral probability is $p = \frac{a-d}{u-d}$ where $a = e^{(r-q)\Delta t}$, while for a stock without dividends $a = e^{r\Delta t}$
This is not unexpected, because in a risk neutral world (in which we are when we use risk neutral probabilities) the total (risk neutral) return should be the risk free rate. When dividends are paid, the total return is the return from the portfolio and from the dividends. As the latter are q, the former must be $r - q$ for a stock with a continuous dividend yield.
For a stock without dividends $q = 0$ so $a = e^{r\Delta t}$ and we find the value of a the we found before.

7.2.2 Black and Scholes model.

If we apply in Black and Scholes a similar reasoning then we will find a differential equation for all derivatives paying dividends:

$$\frac{\partial G}{\partial S}(r-q)S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 = rG$$

The results are also very similar in this case (as before we change the European call and put notation to c and p):

$$c = EC_0 = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

$$p = EP_0 = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

where

- $d_1 = \frac{ln(S_0/K)+(r-q+\sigma^2/2)T}{\sigma\sqrt{T}}$
- $d_2 = d_1 - \sigma\sqrt{T}$

7.3 A special case: options on currencies.

7.3.1 Binomial model.

An option on a currency is like an option on a stock with a continous dividend, the underlying stock is the curreny and the dividend yield is the interest rate on that underlyinh currency, so we apply the same trick as for forwards on currencies.

So we find that the risk neutral probability is $p = \frac{a-d}{u-d}$ where $a = e^{(r-r_{currency})\Delta t}$.
This is not unexpected, because in a risk neutral world (in which we are when we use risk neutral probabilities) the total (risk neutral) return should be the risk free rate. When interests on the currency are paid, the total return is the return from the portfolio and from the interests on the currency. As the latter are $r_{currency}$, the former must be $r - r_{currency}$ for a stock with a continuous dividend yield.

7.3.2 Black and Scholes model.

If we apply in Black and Scholes a similar reasoning then we will find a differential equation for all derivatives paying dividends:

$$\frac{\partial G}{\partial S}(r-q)S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 = rG$$

The results are also very similar in this case (as before we change the European call and put notation to c and p):

$$c = EC_0 = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

$$p = EP_0 = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

where

- $d_1 = \frac{ln(S_0/K)+(r-q+\sigma^2/2)T}{\sigma\sqrt{T}}$
- $d_2 = d_1 - \sigma\sqrt{T}$

7.4 A special case: options on futures.

7.4.1 Binomial model.

A special case is when in section 6.2, the underlying is a future or a forward. Assume that at $t = 0$ the forward price is F_0. Note that F_0 is the price in the contract that makes the value of the contract zero at $t = 0$, so in such a contract the contract price is F_0. In a period Δt the forward price (in the contracts) may rise to F_u or decrease to F_d. What is the value EC_0 of a call on that future contract at $t = 0$?

We proceed in a similar way: construct a portfolio with (a) a loan B, write Δ futures.

- If the price goes up to F_u, then we have in $t = \Delta t$ Δ contracts that have a contract price F_0 while the price is now F_u, so the value is $(F_u - F_0)\Delta$, we also have a loan, with interest so $Be^{r\Delta t}$. So te value is $(F_u - F_0)\Delta + Be^{r\Delta t}$;
- If the price goes down to F_d then similarly, the value of the portfolio is $(F_d - F_0)\Delta + Be^{r\Delta t}$

So at $t = \Delta t$ this portfolio is equivalent to our call option if in these two cases these values are resp. EC_u and EC_d:

$$\begin{aligned}(F_u - F_0)\Delta + Be^{r\Delta t} &= EC_u \\ (F_d - F_0)\Delta + Be^{r\Delta t} &= EC_d\end{aligned}$$

Subtract both equations and you find

$$\Delta = \frac{EC_u - EC_d}{F_u - F_d}$$

$$B = e^{-r\Delta t}\left(EC_u\frac{1-d}{u-d} + EC_d\frac{u-1}{u-d}\right)$$

Then again, similar as in section 6.2 we have $EC_0 = \Delta \times 0 + B$, where the value zero occurs because the value of the call is zero (F_0 is the price in the contract, set such that the value of the contract is zero at $t = 0$!).

So we find that the value of our call is

$$EC_0 = e^{-r\Delta t}\left(EC_u\frac{1-d}{u-d} + EC_d\frac{u-1}{u-d}\right)$$

or if we define the risk neutral probability $\tilde{p} = \frac{1-d}{u-d}$

Property 34. *The value of a call on a future is*

$$EC_0 = e^{-r\Delta t}\left(EC_u\tilde{p} + EC_d(1-\tilde{p})\right)$$

This call can be hedged by at the same time writing $\Delta = \frac{EC_u - EC_d}{F_u - F_d}$ *futures.*

7.4.2 Black and Scholes model.

The changes to the formulas of Black and Scholes can be seen by looking at an option on a future as being an option on an underlying stock with a dividend rate equal to the risk free rate r. This can be ween if we compare the risk neutral probbaility above with the one for options on stock with a dividend yields, indeed, if $r = q$ then $e^{(r-q)\Delta t} = e^0 = 1$.

So in the Black-Scholes formulas for a stock with a dividend we have to replace S_0 by F_0 (because the underlying is the future) and q by r, these are know as "Black's formulas", we have:

$$\frac{\partial G}{\partial S}(r-r)S + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 = rG$$

$$c = EC_0 = F_0 e^{-rT}N(d_1) - Ke^{-rT}N(d_2)$$

$$p = EP_0 = Ke^{-rT}N(-d_2) - F_0 e^{-rT}N(-d_1)$$

where

- $d_1 = \frac{ln(S_0/K)+(r-r+\sigma^2/2)T}{\sigma\sqrt{T}}$
- $d_2 = d_1 - \sigma\sqrt{T}$

Note that F_0 is the value at the day you compute c.

Black's model is reasonable for stocks, currencies, indices and commodities, but not for interests.

7.5 Discrete dividends

We have seen the case where the stock yields a continuous dividend that is a percentage of its value. This is an abtraction of reality because this continous yield divident does never occur. However, it is a very good approximation for shares that periodically pay dividends and the share is far from its maturity, i.e. for a long-life option that periodically pays dividends this works well. .

Now we look at what happens when there is a dividend payout at certain (but limited number) discrete days. In that case, the valued of the stock before the dividend payout is S and just after it it is called S_{ex} i.e. ex-dividend, because the value of the stock has decreased because of this dividend payout.

7.5.1 Black-Scholes with discrete dividends.

The stock price today S_0 (i.e. when you compute the BS-formula) is used to pay the dividends at the discrete times later on, so it has two parts:

Riskless part: a part that will be used to pay the dividends (and this is of course equal to the present value of these dividends that will be paid, i.e. $D^* = PV(D_1, D_2, \dots) = \sum_{i=1}^{n} D_i e^{-r(t_i - t_0)}$) and when we assume that the dividends are known then this is riskless;

Obviously we only take dividends into account that are paid before the option matures !

Risky part: This is the "ex dividend part", the part that remains after paying the dividends. This is uncertai (risky) because it depends on the stock price in the future time.

$$S_0 = S_{ex} + D^*$$

The Black Scholes formula may only be applied at the risky part $S_{ex} = S_0 + D^*$. We also must take the volatility of S_{ex} into account in stead of the volatility of S_0 because in S_0 there is a "non-volatile" part D^* !

So in the Black-Scholes formula you replace:

- $S_0 \rightarrow S_0 - D^* = S_0 - PV(D_1, D_2, \dots)$
- $\sigma \rightarrow \sigma_{ex} = \sigma_S \frac{S_0}{S_0 - D^*}$

And the Black-Scholes formula for options with discrete dividends becomes:

$$c = EC_0 = (S_0 - D^*)N(d_1) - Ke^{-rT}N(d_2)$$

$$p = EP_0 = Ke^{-rT}N(-d_2) - (S_0 - D^*)N(-d_1)$$

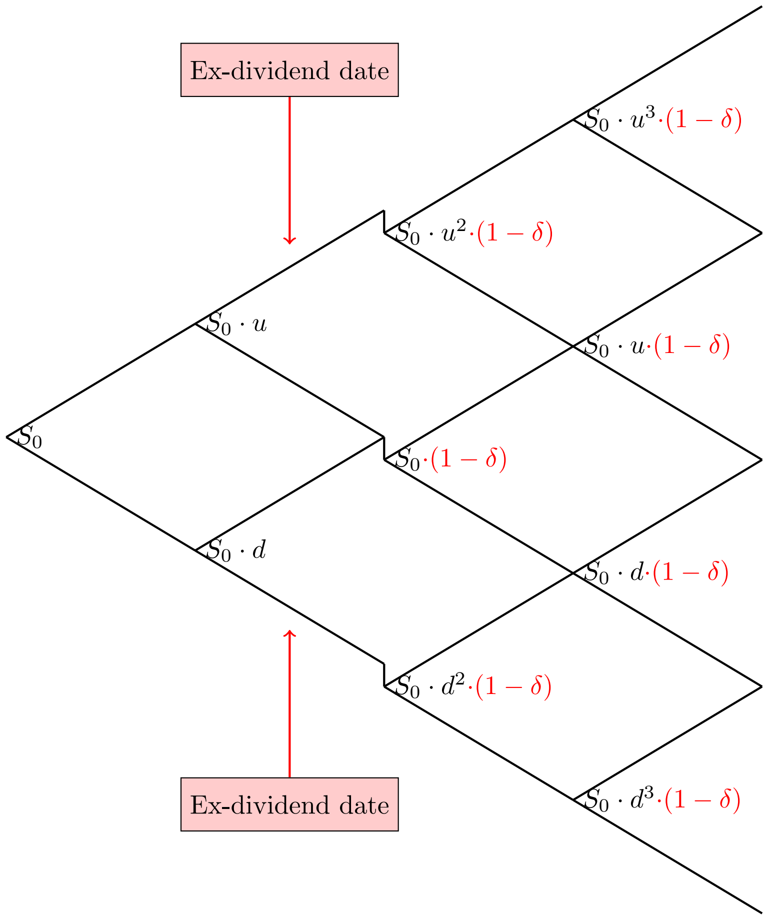
where

- $d_1 = \frac{ln((S_0 - D^*)/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$

- $d_2 = d_1 - \sigma_{ex}\sqrt{T}$
- $D^* = PV(D_1, D_2, \dots) = \sum_{i=1}^{n} D_i e^{-r(t_i - t_0)})$, Only D_i that occur during the life time of the option are taken into account !

7.5.2 Binomial model with discrete dividends.

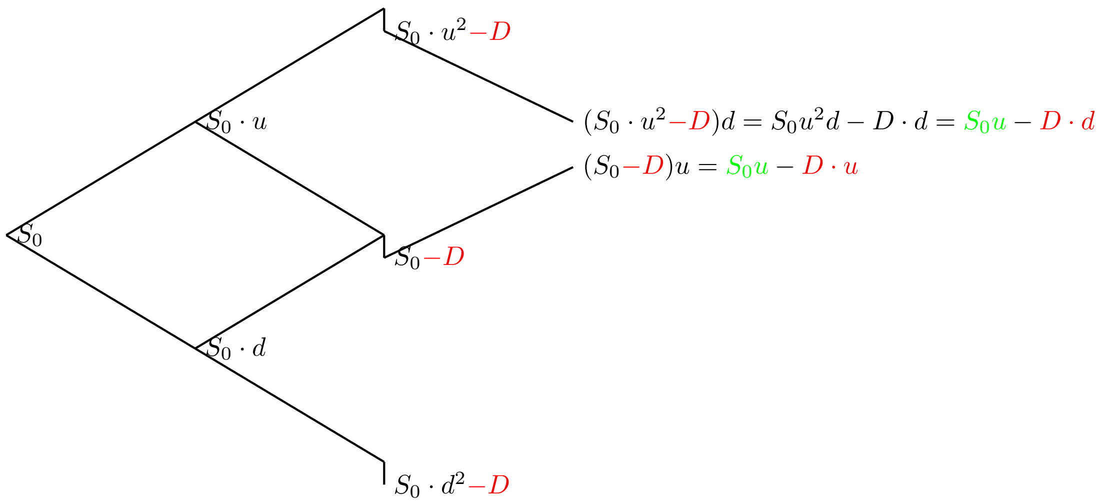
The dividend is a percentage δ of the stock price: In this case the stock price tree looks like this (dividend paid between Δt and $2\Delta t$):



The procedure is similar to the Binomial-tree procedure without divdends, but you have to use the above tree for the stock prices !

The dividend is a dollar amount: This case is a bit more complex because of the following fact: note that in all the trees, the "arrows" in the next step come together, this is because we always have $d = \frac{1}{u}$, such that $u \cdot d = 1$.

If, in the third level of the tree above we have a dollar amount D for the dividend then we have a tree like (note that, see supra $d = 1/u$):



And the arraow at the next step do no longer connect.

as a consequence the number of nodes in the tree increases and the algorithm becomes much more complex !

An alternative procedure should be used, similar to what was done for Black-Scholes with discrete dividends:

- Start the tree with $S_0^* = S_0 - D^*$, where D^* is the present value of all the dividends (that are paid before maturity of the option)
- Build the tree "as usual", but with S_0^* at the start;
- Add D^* to the stock prices before the ex-dividend date

7.6 American options.

An American call (long) without dividends will never be exercised early (see the section on Model independent properties of options).

An American call (long) without dividends may be exercised early but only just before the dividend payment day (names ”just before ex-day”).

An American put (long) can be exercised early (see the section on Model independent properties of options), it must be computed , for examples see exercises !

7.7 Warrants

Warrants are a special kind of options. They are not traded on an exchange. Mostly a company that issues a bind attaches warrants to the bind in order to make the bond more attractive. A warrant is namely an option on a number of shares of the company that issues the warrant.

Let us analyse a warrant that gives the right to buy g shares of the company (so it is similar to a call). Ans assume that the company issues M such warrants at an exercise price K.

When the warrants are exercised the company will create additional shares to fulfill the obligations in the warrant, so, if all warrants are exceuted, $M \times g$ shares will be issued and an amount $M \times K$.

So if the value of the company at the moment the warrant is executed is $V(T)$ and at that moment is has N shares. Then after the warrant being exercised, there will be N shares $M \cdot g$ new shares and the value of the company will be $V(T) + M \cdot K$. So the price per share $P_{after} = \frac{V(T)+M \cdot K}{N+M \cdot g}$

So the call has a value per share: $max(\frac{V(T)+M \cdot K \cdot g}{N+M \cdot g} - K, 0)$, similar to a long call position.

We manipulate this a bit:

$$
\begin{aligned}
P_{after} &= \frac{V(T) + M \cdot K \cdot g}{N + M \cdot g} - K \\
&= \frac{V(T) + M \cdot K \cdot g - K(N + M \cdot g)}{N + M \cdot g} \\
&= \frac{V(T) - K \cdot N)}{N + M \cdot g} \\
&= \frac{N}{N + M \cdot g}(V(T)/N - K)
\end{aligned}
$$

Each warrants gives you the right to receive g stocks.

So the for g warrants the value of the call is $g \times max(\frac{V(T)+M \cdot K \cdot g}{N+M \cdot g} - K, 0) = \frac{g \times N}{N+M \cdot g} max(V(T)/N - K, 0)$

This looks like the formula for $\frac{g \times N}{N+M \cdot g}$ calls on $V(T)/N$ with an exercise price K. Where K is the exercise price in the warrant.

At any time, so even when the warrants are not exercised, the value of the company is $V = N \cdot S + M \cdot W$, where W is the value of the warrant and S the price of one share, because there are N shares and M warrants.

So $V/N = S + \frac{M}{N}W$.

Now we can compute the value of the warrant, being a call option on V/N, using black and scholes where we replace $S \to S + \frac{M}{N}W$, $\sigma_S \to \sigma_{S,warrant}$ and multiply the formula by $\frac{g \times N}{N + M \cdot g}$

Then this formula has to be solved for W using numerical techniques.

7.8 The Greeks.

7.8.1 Illustration of hedging.

Assume that a financial institution has sold (i.e. short position) a call option, then, see the payoff diagram of a short call, the financial institution may risk a very big loss if the spot price at maturity T significantly exceeds the strike price in the option. This loss is even unbounded ! (see payoff).

Therefore the financial institution may wish to "eiminate" this risk, i.e. hedge the position of this short call. Un unhedged position is also called a *naked position*.

It could also take in a *covered position* by buying the same amount of stock as is in the contract. This works well if the option is exercised, but in the other case this may also lead to large losses.

So how can we hedge our risk ?

7.8.2 Delta hedging.

When we determined the value of an option, we constructed a replicating portfolio, consisting of Δ stocks and one call option. This Δ was chosen such that, whatever was the change in the price of the underlying asset, the portfolio had the same value, in other words, Δ was the amount of underlying stock that could compensate for the risk of the call option. It was chosen such that the value of the call combined with Δ stocks

Definition 23. *The Delta of a derivative is the change in value of the derivative for a unit change in value of the underlying stock:* $\Delta = \frac{\partial F}{\partial S}$.

If I have a portfolio of derivatives all with the same underlying asset, e.g. a quantity w_1 *of derivative 1,* w_2 *of derivative 2, ..., then the value of the portfolio is* $\Pi = w_1 F_1 + w_2 F_2 + \cdots + w_n F_n$, *so if we take the derivative we find that* $\Delta = w_1 \frac{\partial F_1}{\partial S} + \cdots + w_n \frac{\partial F_n}{\partial S} = w_1 \Delta 1 + w_2 \Delta_2 + \cdots + w_n \Delta_n$

If you graph the value as a function of S*, then the* Δ *is the slope of the tangent in the point that represents the current situation.*

Property 35. *We have already seen that the* Δ *of a forward is equal to 1. Because of this, a forward is "in terms of* Δ*-hedging equal to stock". Indeed,* Δ *tells you how much stock (or forwards) you need to hedge a derivative position.*

If you look at the formula of Black-Scholes for a European Call without dividends, then $\Delta = \frac{\partial c}{\partial S} = N(d_1)$, and for a European put $\Delta = \frac{\partial p}{\partial S} = N(-d_1)$

When we proved the equality of forward and futures prices (in the contract) we have seen that the delta of a future is $\Delta = e^{r\Delta t}$. If r is small and Δt is small then this is very close to one, Because of this, a future is "in terms of Δ-hedging almost equal to stock".

Remark 7.1. *Note that for some derivatives (e.g. a forward) the Δ is constant and does not change in time, so you can define the hedge at the beginning of the contract and then keep it. This hedging strategy is called "hedge and forget" or "static hedging".*

In other cases (call and put option (note that d_1 depends on σ, on S), future) the hedge has to be adapted every day, this is called a "dynamic hedge".

Also see his slide 39

Example 7.1. Use of Δ-hedging A financial institution has shorted 20 calls (each call has a quantity of 100). It is given that the Δ of this call is 0.52.

How can you hedge this ?

For each shorted stock (100 per call) you can hedge it by bying $\Delta = 0.52$ underlying stocks. So to hegde the 20 calls you need $20 \times 100 \times 0.52 = 1040$ units of stock .

Doing this, the Δ of the portfolio is $20 \times 100 \times 0.52 - 1040 \times 1$ (the delta of a stock is 1 of course). So Δ-hedging makes the Δ of your portfolio equal to zero.

7.8.3 Gamma hedging, Theta hedging.

Definition 24. *We have seen that the Δ of a prtfolio can change, and this leads to "dynamic delta hedging". The Γ of the portfolio (or derivative) is the change of Δ with S, i.e. $\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 \Pi}{\partial S^2}$*

Note that Γ has to do with the concavity/convexity of the value of Π (see slide 45)

Definition 25. *The Θ of the portfolio (or derivative) is the change of its value with time t, i.e. $\Theta = \frac{\partial \Pi}{\partial t}$*

Note that (Taylor), for the value of a portfolio we have

$$\Delta\Pi = \frac{\partial \Pi}{\partial S}\Delta S + \frac{\partial \Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2 \Pi}{\partial t^2}\Delta t^2 + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S \partial t}\Delta S \Delta t + \dots$$

If our portfolio is "delta-hedged" then $\frac{\partial \Pi}{\partial S} = 0$, so we find

$$\Delta\Pi = \overbrace{\frac{\partial \Pi}{\partial S}}^{0}\Delta S + \overbrace{\frac{\partial \Pi}{\partial t}}^{\Theta}\Delta t + \overbrace{\frac{1}{2}\frac{\partial^2 \Pi}{\partial S^2}}^{\Gamma}\Delta S^2 + \frac{1}{2}\frac{\partial^2 \Pi}{\partial t^2}\Delta t^2 + \frac{1}{2}\frac{\partial^2 \Pi}{\partial S \partial t}\Delta S \Delta t + \dots$$

If we take very small Δt then we can forget the second order terms in Δt, so

$$\Delta\Pi = \Theta\Delta t + \Gamma\Delta S^2$$

for a Δ-hedged portfolio.

Substituting these definitions in the Balck-Scholes differential equation we see that the condition for risk-neutrality is

$$\Theta + r \cdot S \cdot \Delta + \frac{1}{2} \cdot \sigma^2 \cdot S^2 \cdot \Gamma = r \cdot F$$

See slides 46-54

8 NEW:Value at Risk (VaR)

We have to do this TOGETHER to make it complete
look at his slides after reading this section

8.1 Definition of VaR

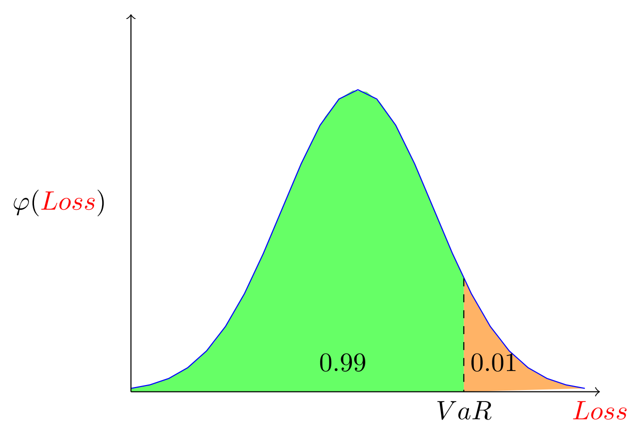
Value at Risk is a measure of market risk. So if you own bonds, securities, equity, ... then their prices can change from day to day. Hopefully they rise and then you have a profit, but sometimes prices decrease and then you incur losses. Value at Risk is a measure of how much your loss may be in exceptional cases, so it tells you how bad things can go.

Definition 26. *The value at Risk (VaR) is a measure of market risk, it is an amount that you can lose over a given time period and with a given confidence level.*

So VaR has two parameters; the length of the time period and the confidence level. The length of the time period is typically one day or ten days. The confidence level is typically between 95% and 99%.

Let's take an example and compute the VaR for a time period of one day and a confidence interval of 99%, so we want to find the amount that we can lose in one day with a confidence level of 99%. In order to find this, we will need the distribution of the losses of our portfolio, this means that we need, for each possibe value of the loss in one day (note that a profit is a negative loss) the probability of occurrence of the loss, which is the loss distribution.

Let us assume that this loss distribution looks like in the figure below. So on the horizintal axis we have the possible values for the loss, and vertically we have the density of the losses (e.g. a histogram):



From the picture we can see what is meant by VaR for one day and a confidence level of 99%: It is the value of the one-day loss that is not exceeded in 99% of the cases, or the value of the one-day loss that is exceeded in 1% of the cases. In formulas we define VaR as the value of the one-day loss for which it holds that $P(Loss \leq VaR) = 99\%$.

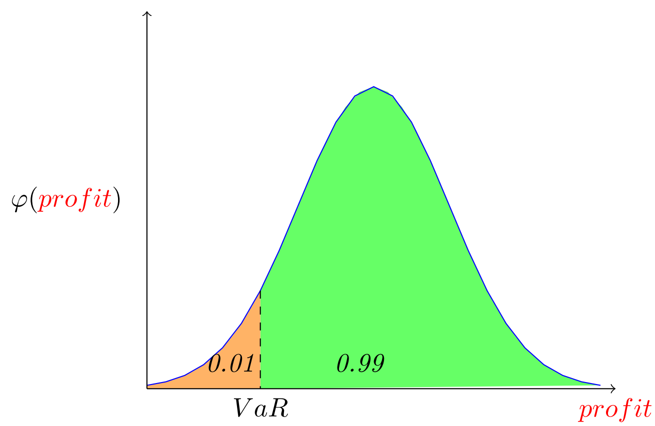
Remark 8.1. *Note that VaR is an amount (in euro) !*

Remark 8.2. *Note that the loss can be higher than VaR, every value above VaR is a loss higher than VaR.*

This is sometimes seen as a disadvantage of VaR, we do not know the maximum loss and if the right tail of the distribution is fat than there could be a reasonable chance that the loss is higher than VaR (an example on slide 13).

Because of this other meaures were invented, one often, used is the "expected shortfall". It is similar to VaR but it does not only take the "border value" into account, but the expected value (=the mean) of all the losses in the left tail of the profit distribution. So it takes into account all profits smaller than VaR (taking into account their tail probability).

Remark 8.3. *Note that a negative profit is a loss and vice versa, so profit=-Loss ! The graphs in the slides of FRM are not the loss distribution but the distribution of the profits. The profit distribution is the mirror of the loss distribution (when the mean is zero, which is usaually assumed in VaR). This is shown below; in this case the VaR is the value of the profit that will be exceeded in 99% of the cases, in formula form: VaR is the value of the profit for which* $P(profit \geq VaR) = 99\%$.



8.2 Example for a normal profit distribution.

If the distribution of the profits is normal then we can easily compute the values of VaR using a table or with a calculator. Assume that we want to compute the VaR over a one-day period and assume that the profit distribution is a normal distrubtion. In VaR calculation it is reasonable to assume that the mean is zero, and the standard deviation we have to estimate from data (see later).

So the profits of our portfolio are then $profit \sim N(\mu = 0, \sigma = s_P)$, where s_P is an estimate of the standard deviation of our losses. Note that s_P is a money amount !

From the normal distribution we know:

- $P(profit \leq -2.3263479 \times s_P) = 0.01$
- $P(profit \leq -1.959964 \times s_P) = 0.025$
- $P(profit \leq -1.6448536 \times s_P) = 0.05$
- $P(profit \leq -1.2815516 \times s_P) = 0.1$

So, as an example, in this case the VaR for one-day horizon and with 99% confidence is $-2.3263479 \times s_P$.

Note however that a normal distribution is only an assumption and it does not hold in reality because the normal distribuion has thin tails, while the profit distribution may have fat tails (see slide 13)

Look at the example on slide 10 !

8.3 Finding the loss distribution.

From this definition if follows that, if we know the loss distribution, we can compute the VaR. The problem is of course to find that loss-distribution.

There are three ways to get an "estimate" of the loss distribution:

- Historical VaR; uses the historical price changes to find the profit (loss) distribution;

  So its characteriscs are:

  - uses data from the past and assumes the future will be like the past;
  - no theoretical distribution is assumed
  - does not assume a linear link between the change in value and a change in the risk factor

- The variance-covariance method; is based on the assumption that the assets in our portfolio follow a multi-variate normal distribution.

  So its characteriscs are:

  - uses data from the past and assumes the future will be like the past;
  - a normal distribution is assumed
  - assumes a linear link between the change in value and a change in the risk factor

- Monte Carlo simulations; one choses an appropriate theoretical distribution (that can be more complex than normal) and randomly generates scenarios.

  So its characteriscs are:

  - uses data from the past and assumes the future will be like the past;

- a theoretical distribution is assumed, it can be another distribution than normal
- does not assume a linear link between the change in value and a change in the risk factor

8.3.1 Main idea behind VaR estimation.

We have a portfolio that contains different types of assets (e.g. bonds, shares of equity, options, ...) let's say $A_1, A_2, \ldots, A_n$ are the values of these assets.

Of each asset our portfolio contains a certain quantity, let's call these quantities the weights $w_1, w_2, \ldots w_n$.

So the value of our portfolio is (we use a time argument because the quantities change in time)

$$V_P(t) = w_1(t)A_1(t) + w_2(t)A_2(t) + \cdots + w_n(t)A_n(t)$$

We could go one step further and define "underlying risk factors", e.g. the price of a bond depends on the interest rate, ... so the price of an asset depends in these factors $F_1, F_2, \ldots F_n$, i.e. $A_i(t) = f(F_1, F_2, \ldots, F_k)$, where the F_j also depend on thime. So indirectly our portfolio vaue depends (is a function of) on these risk factors (one of the Fs could be interest rate) !

$$V_P(t) = \phi(F_1(t), F_2(t) \ldots, F_n(t))$$

If the function ϕ simplifies to a linear function then we say that "linearity is assumed":

$$V_P(t) = \alpha_1 F_1(t) + \alpha_2 F_2(t) + \cdots + \alpha_n F_n(t))$$

8.3.2 Historical VaR

We work with the formula $V_P(t) = w_1(t)A_1(t) + w_2(t)A_2(t) + \cdots + w_n(t)A_n(t)$, so with the asset values and with the quatities of each asset.

You need data on e.g. 501 one-day losses for each asset.

With this you can compute 500 growth rates from one dat to the next.

Apply these 500 growth rates to the most recent values and compute V_P for each of the 500 results.

Make a histogram of these 500 V_P

Take (for 99% VaR for one-day horizon) the fith worst value. (1% of 500 is 5)

The method also works for the formula $V_P(t) = \alpha_1 F_1(t) + \alpha_2 F_2(t) + \cdots + \alpha_n F_n(t))$ when you have historical data on the F_j.

8.3.3 Variance-covariance

Here we work with the formula $V_P(t) = \alpha_1 F_1(t) + \alpha_2 F_2(t) + \cdots + \alpha_n F_n(t))$, so with the underlying risk factors.

Note that linearity is assumed. Moreover it is assumed that all factors F_j are multivariate normal, so normality is assumed. You also need to know all the (linear) links between each assets and the factors F_j.

The you estimate the var-covar matrix of the factors. If the F_j are multivariate normal and you have the var-covar matrix, then you can compute the distribution of $V_P(t) = \alpha_1 F_1(t) + \alpha_2 F_2(t) + \cdots + \alpha_n F_n(t))$ and thus find the VaR.

In the linear case V_P has a normal distribution and its standard deviation can be computed form the var-covar matrix and the α's, so you can apply the "normal distribution case" supra.

8.3.4 Monte-carlo simulation

Similar to the previous but you assume other distributions and then simulate from these. Then you get (simulated) values for the Fs. With this you can compute V_P even for the more complicated ϕ !

8.4 VaR backtesting.

When you compute the VaR, you make a lot of assumptions; that the past will extend to the future, about the distribution, about the functional link between the loss and the risk factors, ... So what you find as VaR will depend on whether these assumtpions are fulfilled in the real world.

So what you typically do is to compute the VaR and after that, with other data, you check whether on this new data the loss is lower than VaR computed in x% of the cases.

You should never use a VaR that has not been backtested !

8.5 VaR and Basel

9 Credit Risk

10 NEW:Case studies

10.1 Forward rate agreement. Exam question Q3.

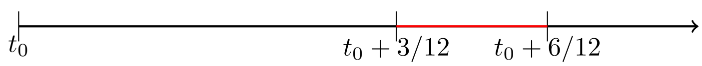
PNC has hedged (in the past apparently) its interest rate expsoure, that was a floating rate on an amount of 100 mio EUR. They used an FRA for this. When the floatig rate was created (in the past) the fixed rate was 3%. This means that at $t = 0$ (somewhere in the past) the fixed rate of 3% made the initial value of ths FRA zero (see section on FRA).

What does this hedge mean ? Well PNC knew that it would need a loan between 1/4/2010 and 30/6/2010 and, because this is in the future, it can't know the (floating) rate at the initiation of the FRA (note that an FRA is a swap with just one cash flow !). It does not like this uncertainty and wanted to find a way to pay a fixed and known rate for that period. So th "to pay floating rate" for that period was compensated by a "receive floating" (exactly opposite to the one "to pay") and "pay fix". So the FRA pays a fixed rate of 3% (in exchange for the floating rate of x%). Combining its original loan with this FRA results in a fixed rate payment because the "to pay float" and the "receive float in the FAR" net to zero !

The creation was in the past and today we are at $t = t_0 = 31/12/2009$. Because the expected rates for the period 1/4/2010-30/06/2010 have chenged compared to those at creation, the FRA's value may be different from zero at t_0 !

So we want to compute the value of a (pay fix) FRA at $t_0 = 31/12/2009$ for a loan over the period $t_1 = 1/4/2010 = t_0 + 3/12$ to $t_2 = 30/06/2010 = t_0 + 6/12$.

It is said that compounding is simple and we don't have to care about daycount conventions, i.e. it is 30/360.



In the section on FRA we saw the formula for the value of the FRA.

$$V = L(f^{t_0}_{t_0+3/12,t_0+6/12} - R_{fra})(t_0 + 6/12 - (t_0 + 3/12))DF(t_0, t_0 + 6/12)$$

We use simple compounding, therefore $DF(t_0, t_0+6/12) = \frac{1}{1+R_{t_0,t_0+6/12}(t_0+6/12-t_0)}$

So

$$\begin{aligned} V &= L(f^{t_0}_{t_0+3/12,t_0+6/12} - R_{fra})(\cancel{t_0} + 6/12 - \cancel{t_0} - 3/12))\frac{1}{1 + R_{t_0,t_0+6/12}(\cancel{t_0} + 6/12 - \cancel{t_0})} \\ &= \overbrace{L}^{\text{100 mio EUR}}(f^{t_0}_{t_0+3/12,t_0+6/12} - \overbrace{R_{fra}}^{3.00\%})(3/12))\frac{1}{1 + \underbrace{R_{t_0,t_0+6/12}}_{\text{see table,3.00\%}}(6/12)} \end{aligned}$$

So if we find $f^{t_0}_{t_0+3/12,t_0+6/12}$ then we just have to put the numbers in the calculator !

We can find this forward rate using the schema in figure 1, but we have to use simple compounding, so we have

$$(1+R_{t_0,t_0+3/12}(t_0+3/12-t_0))(1+f^{t_0}_{t_0+3/12,t_0+6/12}(t_0+6/12-(t_0+3/12))) = (1+R_{t_0,t_0+6/12}(t_0+6/12-t_0))$$

so we find

$$f_{t_0+3/12,t_0+6/12} = \frac{1}{(6/12-3/12)}\left(\frac{(1+\overbrace{R_{t_0,t_0+6/12}}^{\text{see table,3.00\%}}(6/12))}{(1+\underbrace{R_{t_0,t_0+3/12}}_{\text{see table,2.50\%}}(3/12))}-1\right)$$

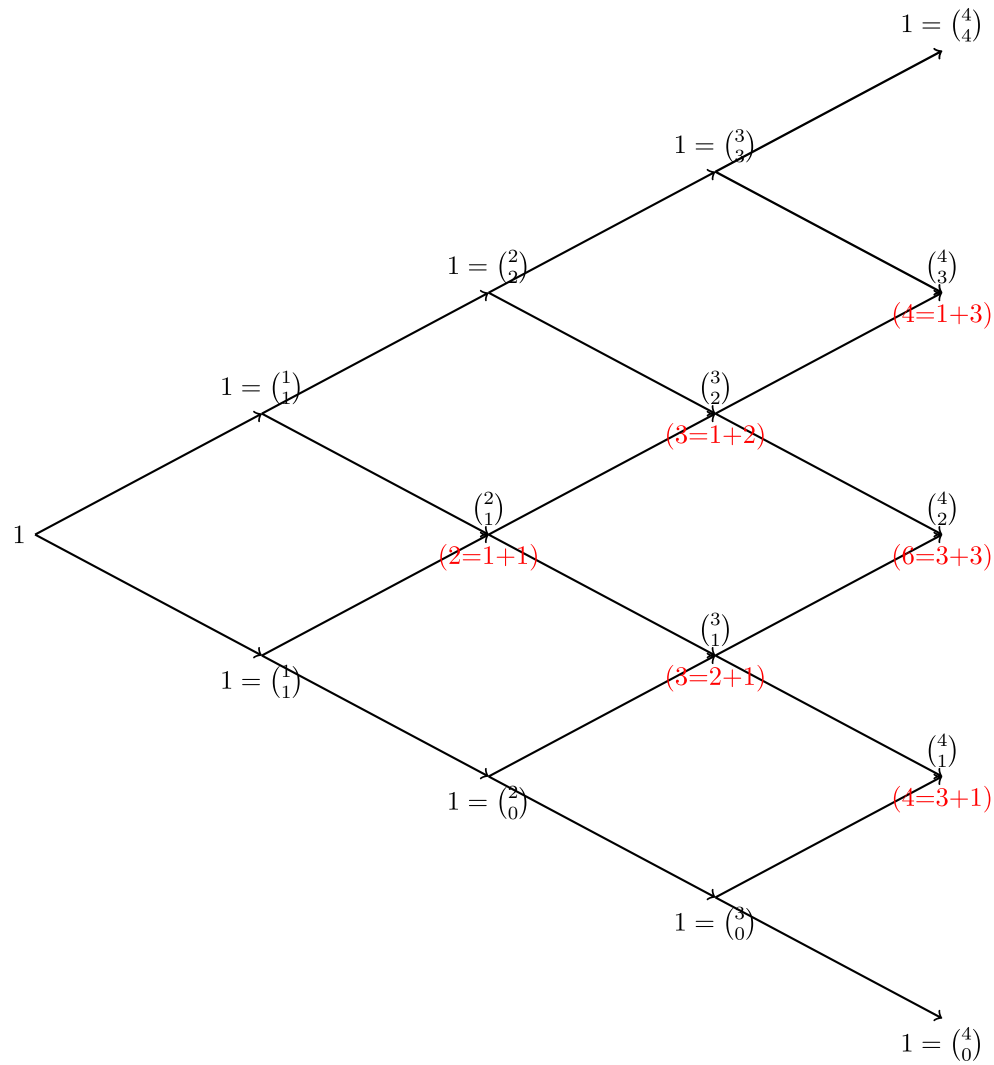
With all this we can compute $f_{t_0+3/12,t_0+6/12}$ and with that we can (see supra) compute V.

Part b) is still TO DO

10.2 Binomial coefficients (Pascal's triangle).

If you have to compute the Binomial probabilities $\binom{n}{k}$ then you can use the so-called triangle of Pascal. At the outer nodes you always have '1' and in each node you just sum up the values of the nodes that arrive in that node. So e.g. $\binom{2}{1} = 2$ because the two arrows that arrive in $\binom{2}{1}$ have values 1 and 1, so $\binom{2}{1} = 1 + 1$. Similarly, $\binom{4}{2} = 3 + 3$

So you compute the tree by starting at the left.



11 Exercises.

11.1 Bonds and interest rates.

11.1.1 Q1.1

An investor receives 5616 in 3y, in return for an investment of 5400 today, compute percentage per annum with.

- Annual compounding: We should apply the formula 1 for discrete compounding, with $m = 1$, $N_0 = 5400$, $N_{0+3} = 5616$ and $\Delta t = 3$ and solve for $R^{(1)}_{0,x}$:

$$5616 = 5400 \left(1 + \frac{R^{(1)}_{0,x}}{1}\right)^{1\times 3}$$

  If follows that $R^{(1)}_{0,x} = \sqrt[3]{\frac{5616}{5400}} - 1 = 0.0131594$

- semi-annual; use $m = 2$ and you get

$$5616 = 5400(1 + R^{(2)}_{0,x}/2)^{2\times 3}$$

  If follows that $R^{(2)}_{0,x} = 2\sqrt[6]{\frac{5616}{5400}} - 1 = 0.0131164$

- monthly, take $m = 12$ and you get $R^{(12)}_{0,x} = 12\sqrt[36]{\frac{5616}{5400}} - 1 = 0.0130807$

- daily, take $m = 365$ and you get $R^{(365)}_{0,x} = 365\sqrt[2\times 365]{\frac{5616}{5400}} - 1 = 0.0130738$

- For continuous compounding you apply the formula 1 for continuous compounding.

  We have to solve the following equation for $R^c_{0,x}$:

$$5616 = 5400e^{R^c_{0,x}\times 3}$$

  Or $R^c_{0,x} = 1/3ln(5616/5400) = 0.0130736$

11.1.2 Q1.2

Consider following zero coupon rates (ZCR) in annual compounding (AC):

Years	ZCR(AC)
5	1,03%
7	1,11%
8	1,26%
10	1,34%

- Compute the discount factors from these rates: For discount rates we use the fomulas 2. The rates in the above are with annual compounding, so $m = 1$, the Δt is expressed in years and is found in the first column of the table.
  - $DF(0;5)^{(1)} = \frac{1}{(1+0.0103)^5} = 0.9500539$
  - $DF(0;7)^{(1)} = \frac{1}{(1+0.0111)^7} = 0.9256381$
  - $DF(0;8)^{(1)} = \frac{1}{(1+0.0126)^8} = 0.9046834$
  - $DF(0;10)^{(1)} = \frac{1}{(1+0.0134)^{10}} = 0.8753687$
- Convert the rates to continuous compounding: Using formula 1 we have to find $R^c_{t,T}$ that gives the same amount as $(1 + R^{(1)}_{t,T})^{\Delta t}$, so we have to solve the equation $e^{R^c_{t,T}\Delta t} = (1+R^{(1)}_{t,T})^{\Delta t}$ after taking the Δt-root on both sides we have

$$e^{R^c_{t,T}} = (1 + R^{(1)}_{t,T})$$

  for $R^c_{t,T}$ using the values in the first column for T and the ones in the second column for $R^{(1)}_{t,T}$.

  So from the table we find:
  - $R^c_{t,5} = ln(1 + 0.0103) = 0.0102473$
  - $R^c_{t,7} = ln(1 + 0.0111) = 0.0110388$
  - $R^c_{t,8} = ln(1 + 0.0126) = 0.0125213$
  - $R^c_{t,10} = ln(1 + 0.0134) = 0.013311$
- Compute the discount rates using the continuous compounding formula. We have to apply formula 2 using the previously found rates with continuous compounding: For example $DF^{(1)(0;5)} = e^{-0.0102473\times 5} = 0.950054$.
- Compute the forward rate for the 5-10 period in annual compounding and continous compounding.
  - Annual compounding:

You don't have to learn the formulas, just use the schema in figure 1, but using the appropriate discounting frequency.

    $\left(1 + \frac{R^m_{0,T}}{m}\right)^{mT} = \left(1 + \frac{R^m_{0,t}}{m}\right)^{m\cdot t} \times \left(1 + \frac{f^m_{t,T}}{m}\right)^{m(T-t)}$, with $m = 1$.

    We can take the m-th root and have $(1 + \frac{R^m_{0,T}}{m})^T = (1 + \frac{R^m_{0,t}}{m})^t(1 + \frac{f^m_{t,T}}{m})^{(T-t)}$.

    So $(1 + \frac{f^m_{t,T}}{m})^{(T-t)} = \frac{(1+\frac{R^m_{0,T}}{m})^T}{(1+\frac{R^m_{0,t}}{m})^t}$ and therefore:

$$f^m_{t,T} = m \times \left(\sqrt[T-t]{\frac{(1+\frac{R^m_{0,T}}{m})^T}{(1+\frac{R^m_{0,t}}{m})^t}} - 1 \right)$$

The exercise aks for annual, so $m = 1$, we know that $t = 5, T = 10$ and $R_{0,10} = 0.0134, R_{0,5} = 0.0103$. So $f^{(1)}_{5,10} = 0.0165095$

– Continuous compounding:

You don't have to learn the formulas, just use the schema in figure 1, but using the appropriate discounting frequency.

$e^{R^c_{0,T}T} = e^{R^c_{0,t}t} \times e^{f^c_{t,T}(T-t)}$ from which you find that $R^c_{0,T}T = R^c_{0,t}t + f^c_{t,T}(T-t)$, from which it is easy to find $f^c_{t,T}$.

$f^c_{5,10} = \frac{0.1331101-0.0512366}{10-5} = 0.0163747$

11.1.3 Q1.3

A bank quotes you an interest rate of 12.98% p.a. with daily compounding. What is the equivalent rate with

- Continuous compounding: each euro is after one year $(1+0.1298/365)^{365}$ euro. The equivalent rate on continuous compounding is $R^c_{0,x}$ where $e^{R^c_{0,x}} = (1+0.1298/365)^{365}$ or $R^c_{0,x} = ln((1+0.1298/365)^{365}) = 0.1297769$.

- Annual compounding, in two ways: Here we have to find $R^{(1)}_{0,x}$ where $(1+\frac{R^{(1)}_{0,x}}{1})^1 = (1+0.1298/365)^{365}$ or $R^{(1)}_{0,x} = (1+0.1298/365)^{365} - 1 = 0.1385744$

  The second way is by solving $(1+\frac{R^{(1)}_{0,x}}{1})^1 = e^{R^c_{0,x}}$, where $R^c_{0,x}$ is the value found supra.

11.1.4 Q1.4

Zero interest rates with continuous compounding are

Maturity (months)	$R^c_{0,T}$
3	1.2130%
6	1.5134%
9	1.7896%
12	2.1258%
15	2.5698%
18	3.3569 %

- forward rate second quarter; $e^{R^c_{0,0.25}0.25}e^{f^c_{0.25,0.5}0.25} = e^{R^c_{0,0.5}0.5}$, so $f^c_{0.25,0.5} = \frac{R^c_{0,0.5}0.5 - R^c_{0,0.25}0.25}{0.25} = 0.018138$.

  Similar: $f^c_{0.5,0.75} = \frac{R^c_{0,0.75}0.75 - R^c_{0,0.5}0.5}{0.25} = 0.02342$

  Rest in the same way

- Convert to simple annual compounding: $e^{R^c_{0,0.25}0.25} = (1 + R_{0,0.25} \times 0.25)$ ($\Delta t < 1$)), or $R_{0,0.25} = \frac{e^{R^c_{0,0.25}0.25}-1}{0.25} = 0.0121484$

  Similar: $R_{0,0.5} = \frac{e^{R^c_{0,0.5}0.5}-1}{0.5} = 0.0151914$

11.1.5 Q1.5

Consider the following bond prices with their respective coupon payments.

Maturity	Bond price	Coupon	Coupon payment
3 months	97.50	0	NA
6 months	95	0	NA
1 year	90	0	NA
1.5 years	95	8	paid every six months
2 years	100	12	paid every six months

- Bootstrap the continuously compounded interest rates:

  Why don't we use the formula for $\Delta t < 0$ here ?

  The first three have zero coupons and are therefore easy to calculate: for the firts one; you buy it today at 97.5 and receive the face value of 100 in 3 months, so,

  $100 = 97.5e^{R^c_{0,0.25}0.25}$ or $R^c_{0,0.25} = ln(100/97.5)/0.25 = 0.1012712$

  Similar for the second $100 = 95e^{R^c_{0,0.5}0.5}$ or $R^c_{0,0.5} = ln(100/95)/0.5 = 0.1025866$

  Third one: $100 = 90e^{R^c_{0,1}1}$ or $R^c_{0,1} = ln(100/90)/1 = 0.1053605$

  The last two are a bit more complicated because we get intermediate coupon payments. The coupon is 8 per year, but paid every 6 months , so 4 every six months. Therefore the currrent valus of the cash flows for the fourth is

  $95 = 4e^{-R^c_{0,0.5}0.5} + 4e^{-R^c_{0,1}1} + 104e^{-R^c_{0,1.5}1.5}$ where the values of $R^c_{0,0.5}$ and $R^c_{0,1}$ have been computed supra. And where $R^c_{0,1.5}$ must be found from that equation.

  $104e^{-R^c_{0,1.5}1.5} = 95 - 4e^{-0.1025866\times0.5} - 4e^{-0.1053605\times1} = 87.6$

  So $R^c_{0,1.5} = -ln(87.6/104)/1.5 = 0.1144066$

  If you would not have known $R^c_{0,0.5}$ and $R^c_{0,1}$ then you have to use interpolation !

  For the last one we have $100 = 6e^{-R^c_{0,0.5}0.5} + 6e^{-R^c_{0,1}1} + 6e^{-R^c_{0,1.5}1.5} + 106e^{-R^c_{0,2}2}$ or $106e^{-R^c_{0,2}2} = 100 - 6e^{-R^c_{0,0.5}0.5} - 6e^{-R^c_{0,1}1} - 6e^{-R^c_{0,1.5}1.5}$. We already found $R_{0,0.5}, R_{0,1}, R_{0,1.5}$ (hence bootstrap)[17].

  It follows that $106e^{-R^c_{0,2}2} = 100 - 6e^{-0.1025866\times0.5} - 6e^{-0.1053605} - 6e^{-0.1144066\times1.5} = 83.8461538$ or $R^c_{0,2} = -1/2 \times ln(83.8461538/106) = 0.1172277$

[17] bootstrap = schoenaantrekker; je met dus eerst die van 6m en een jaar hebben om daarna die van 1,5 jaar te berekenen, ...

Consider the following bond prices with their respective coupon payments.

Maturity	Bond price	Coupon	Coupon payment
3 months	97.5310	0	NA
6 months	95.0041	0	NA
9 year	92.4271	0	NA
1 years	94.2980	5	paid every 12 months
2 years	97.2578	10	paid every 12 months

In part a) we looked at $t = T$, we can do similar but look at $t = 0$:

$97.5310 = 100e^{-R^c_{0,3/12}\times 3/12}$ from which you find $R^c_{0,3/12}$

$95.0041 = 100e^{-R^c_{0,6/12}\times 6/12}$ from which you find $R^c_{0,6/12}$

$92.4271 = 100e^{-R^c_{0,9/12}\times 9/12}$ from which you find $R^c_{0,9/12}$

$94.2980 = 5e^{-R^c_{0,12/12}\times 12/12}+100e^{-R^c_{0,12/12}\times 12/12}$ from which you find $R^c_{0,12/12}$

$97.2578 = 10e^{-R^c_{0,12/12}\times 12/12} + 10e^{-R^c_{0,24/12}\times 24/12} + 100e^{-R^c_{0,24/12}\times 24/12}$ from which you find $R^c_{0,24/12}$

11.2 Forwards and futures.

11.2.1 Q2.1

A stock trades a 50 euro , interest is compounded continuously at 1.5% for all maturities.

- If I enter a forward purchase of that share for delivery in 9 months, what will be the purchase price at the delivery date ? How much do I pay today ?

  > If you enter a forward contract than the delivery price, or the price in the contract is determined as F_0 like explained in 2.3.1.

  You can learn these formulas by heart, but they are easy to find, the only thing you must keep in mind is that all profits and costs between [0,T] must be taken into account.

  In this example there are no profits nor costs involved in keeping the stock froml [0,T], so we borrow $S_0 = 50$ euro, buy the stock at the spot market (and keep it until $T = 9/12$).

  It is said that compounding is continuosly so $F_0^{EUR/stock} = S_0^{EUR/stock} e^{R^c_{0,T}T}$ where $S_0^{EUR/stock} = 50$ EUR, $R^c_{0,T} = 0.015$ and $T = 9/12$, so we find that :

  $F_0^{EUR/stock} = 50.565676$ EUR/Stock.

- Three months later the share trades at 57 euro and interest rates are 1.75%. What is the fair value of the contract ? Compute this fair value in two ways:

  Here we are not asked for the delivery prince in the contract, but for the fair value of the contract. We again have to use continuous compounding so we have to use the formulas 9 and 10.

The formula 10 must applied to $t = 0$ and $t = 3/12$:

- At $t = 0$ we use the fair value computed above as the contract price, i.e $K = F_0^c = 50.565676$.
- Three months later the fair value of the delivery price should be $F_t = S_t e^{r_t(T-t)}$ where $S_t = 57, r_t = 0.0175, t = 0.25, T = 0.75$ so $F_t^{c,EUR/stock} = 57.5009384$ EUR.

So in the contract the value at T will be 50.565676 but the fair value of the stock at T after 3m is 57.5009384, so the value of the itself at T is 6.9352624 and at $t = 3m$ it should be discounted at the rate 1.75%, so the value is $f_t^c = (F_t^c - F_0^c)e^{-r_t(T-t)} = 6.8748436$

- Repeat the two first questions assuming that you get a dividend of 5 euro after 2 months.

  F_0 has to take the dividend after two months into account, i.e. we can subtract that value from the cost of borrowing the money; because the cost of borrowing is reduced by the amount earnt on the dividend.

  The present value of the dividend payment is $i_0^{[0;T]} = 5e^{-2/12 \times 0.015} = 4.9875156$.

  $F_0 = (S_0 - i_0^{[0;T]})e^{r_0 T} = 45.521734$.

  The is no impact on F_t because that is after three months and there is no dividend between the 3th and the 12th month, only after the second month. So $i_t^{[t;T]} = 0$

  The rest is similar, so we find $f_t = 11.8748436$

11.2.2 Q2.2

1 euro is worth 1.12 USD. Interest rates (simple annual compounding) are:

Maturity (months)	$R_{0,T}^{EUR}$	$R_{0,T}^{USD}$
3	0.25%	1.25%
6	0.5%	1.5%
9	0.75%	1.75%

- Compute the forward exchange rates for these maturities.

  For the procedure to solve the problems with forward exchange rates see exercise 11.6.1.

  1. What is the asset ? What is the currency used for paying ?
     - In the second part of the question it is said that $Q = 1,000,000$ EUR, as Q is the amount of asset, the asset in this exercise is **Asset=1EUR**.
     - The currency used **to pay is the USD**.
  2. What are r, q ?

- r is the interest rate on the money that is used to pay, so $r = R^{USD}$
- q is the profit rate in the asset (1EUR), so $q = R^{EUR}$

3. What is the compounding frequency ?

   In the exercise it is said that this is **simple annual accounting**. So, following the same reasoning as for formula **??** we find $F_t^{s,USD/EUR} = S_t^{USD/EUR} \frac{1+R_t^{USD}\Delta t}{1+R_t^{EUR}\Delta t}$

4. discount rate for computing f_t, compounding frequency and log or short ?

   - To compute f_t we need discount rate, it is the rate for the currency that you use to pay, so R^{USD}
   - The compounding is simple annual compounding
   - we buy so we have a long position

   So we have to use $f_t^{USD/EUR} = (F_t^{USD/EUR} - F_0^{USD/EUR}) \frac{1}{1+R_t^{USD}\Delta t}$.

This would give the following results:

- $T = 3/12$ we find $F_0 = 1.12$ USD/EUR $\times \frac{1+0.0125\times 3/12}{1+0.0025\times 3/12}$ or 1.1227983 USD/EUR
- $T = 6/12$ we find $F_0 = 1.125586$ USD/EUR
- For $T = 9/12$ we find $F_0 = 1.128353$ USD/EUR

- I have a forward contract to buy 1 million euro against 1.20 million dollar for delivery in 2 years from now. The spot rate today is 1.12 USD/EUR. Assume that the euro and dollar interest rates are 1% and 2% respectively (continuous compounding). What is the fair value of the foward contract.

  Here we are asked for the value <u>of</u> the contract f_t, not the value in the contract. For continuous compounding we have to use the formula 9

  Where F_0 is the value in the contract when it was signed, i.e. 1.2 USD/EUR.

  For F_t we apply the formula **??** so $F_t = 1.12 \times e^{(0.02-0.01)\times 2} = 1.1426255$ and then, after discounting we find that $f_t = -0.0551248$.

  ṫextcolorredThe discount rate is the interest rate for the USD because when you look at how formula 9 was derived you have USD. see at the end of section 2.6.1

  As $Q = 1000000$ have to multiply this by that amount and we find

  -55124.8128792

11.3 Swaps.

11.3.1 Q3.1

Read the overview in section 3.4.1. A swap is a contract where you pay a fixed rate and receive a floating rate or vice versa. The cash flows that result from this contract are paid periodically and thus the interests are not compounded.

The floating rates are unknown (because they are in the future), but as in other derivatives, initially the fixed rate is determined such that, initially, the sum of the discounted values of all the cash flows is zero.

This fixed rate is called the swap rate. There is of course a swap rate for each maturity so we have $S_{0,T}$ for different values of T. In order to compute this fixed rate, as just mentioned, we whave to discount, so this swap rate will also depend on the discounting frequency, so we have $S^*_{0,T}$. The fixed swap rate interests are paid periodically, thus tehy are not compounded and these swap rates can not be udes to compute discount factors.

There are two ways to find this fixed rate, called the swap rate,

Using bond valuation techniques: A property shown in the course is that the swap rate makes the discounted value of a bond with a fixed rate and with periodic payments equal to its face value (see section 3.3.3 for detail). We assume annual compounding because of the data in the exercise. Let us call the unknown swap rate $S_{0,4}$ and assule you have a swap where the face value is F.

Then after one year you pay (pay because it is in the exercise) an interest rate $S_{0,4} \times F \times 1$, ($\times 1$ because it is over 1 year) discounted this becomes $S_{0,4} \times F \times DF^{(1)}(0,1)$;

Then after two year you pay an interest rate only on the face value, (so not on the interest in the first period !) $S_{0,4} \times F \times 1$, discounted this becomes $S_{0,4} \times F \times DF^{(1)}(0,2)$;

Then after three year you pay an interest rate only on the face value, (so not on the interest in the first period !) $S_{0,4} \times F \times 1$, discounted this becomes $S_{0,4} \times F \times DF^{(1)}(0,3)$;

Then after four year you pay an interest rate only on the face value, (so not on the interest in the first period !) $S_{0,4} \times F \times 1$, discounted this becomes $S_{0,4} \times F \times DF^{(1)}(0,4)$. After four year you also pay the face value F, so after discounting to today this is $F \times DF^{(1)}(0,4)$

The swap rate is the value of $S_{0,4}$ that makes the sum of these discounted values, i.e. the value of the bond at $t = 0$ equal to its face value (see section 3.4.4) or we have to solve the following equation for $S_{0,4}$:

$$\begin{aligned} F &= F \times (S_{0,4} \times DF(0,1)^{(1)} + S_{0,4} \times DF^{(1)}(0,2) \\ &+ S_{0,4} \times DF^{(1)}(0,3) + S_{0,4} \times DF^{(1)}(0,4) + DF^{(1)}(0,4)) \end{aligned}$$

So we find that $1 = S_{0,4}(0.9800 + 0.9500 + 0.9250 + 0.8900) + 0.8900$ or $S_{0,4} = 0.0293725$

Discounting the cash flows: In this method we sum up the difference cash flows, but in the fixed cash flow we use an unknown value $S_{0,4}$

Obviously, the parties in this contract will take into account the interest rates that are valid at that time. Assume that we know the (annual, discrete) discount factors $DF^{(1)}(0,1) = 0.9800, DF^{(1)}(0,2) = 0.9500, DF^{(1)}(0,3) = 0.9250, DF^{(1)}(0,4) = 0.8900$.

We have cash flows after 1,2,3 and 4 year. We have to find the value of these cash flows at $t = 0$.

- After the first year we have
  - A fixed flow that we pay, so $-S_{0,4} \times F \times 1$ ($\Delta t = 1$) and discounted $-S_{0,4} \times F \times DF^{(1)}(0,1)$;
  - A floating cash flow at an known (today, at $t = 0$, $R_{0,*}$ are all known) floating rate $R^{(1)}_{0,1}$, this known rate can be derived from the discount rates as explained in section 3.4.2: $DF^{(1)}(0,1) = \frac{1}{(1+R^{(1)}_{0,1})^{(1-0)}}$ or $R^{(1)}_{0,1} = \frac{1}{DF^{(1)}(0,1)} - 1 = 0.0204082$
    So the floating cash flow is $+0.0204082 \times F \times 1$ and discounted $+0.0204082 \times F \times DF^{(1)}(0,1)$
  - So the cash flow after 1 year is $-S_{0,4} \times F \times DF^{(1)}(0,1) + 0.0204082 \times F \times DF^{(1)}(0,1)$
- After the second year we have
  - A fixed flow that we pay, so $-S_{0,4} \times F \times 1$ ($\Delta t = 1$) and discounted $-S_{0,4} \times F \times DF^{(1)}(0,2)$;
  - A floating cash flow at an UNKNOWN floating rate $f^{(1)}_{1,2}$, this unknown rate can be derived from the discount rates as explained in section 3.4.3: $\frac{1}{DF^{(1)}(0,1)} \frac{1}{DF^{(1)}(1,2)} = \frac{1}{DF^{(1)}(0,2)}$ or $DF^{(1)}(1,2) = \frac{DF^{(1)}(0,2)}{DF^{(1)}(0,1)}$.
    Since $DF^{(1)}(1,2) = \frac{1}{(1+f^{(1)}_{1,2})^{2-1}}$ we have $(1+f^{(1)}_{1,2})^{2-1} = \frac{DF^{(1)}(0,1)}{DF^{(1)}(0,2)}$ of $f^{(1)}_{1,2}) = \frac{DF^{(1)}(0,1)}{DF^{(1)}(0,2)} - 1 = 0.0315789$
    We <u>assume</u> that the floating rate after 1 year will be $f^{(1)}_{1,2}$). So the cash flow that results is $+0.0315789 \times F \times 1$ and discounted $+0.0315789 \times F \times DF^{(1)}(0,2)$
  - So the cash flow after 2 year is $-S_{0,4} \times F \times DF^{(1)}(0,2) + 0.0315789 \times F \times DF^{(1)}(0,2)$
- After the third year we have
  - A fixed flow that we pay, so $-S_{0,4} \times F \times 1$ ($\Delta t = 1$) and discounted $-S_{0,4} \times F \times DF^{(1)}(0,3)$;

- A floating cash flow at an UNKNOWN floating rate $f^{(1)}_{1,2}$, this unknown rate can be derived from the discount rates as explained in section 3.4.3: $\frac{1}{DF^{(1)}(0,2)}\frac{1}{DF^{(1)}(2,3)} = \frac{1}{DF^{(1)}(0,3)}$ and in a similar way we find that $f^{(1)}_{2,3}) = \frac{DF^{(1)}(0,2)}{DF^{(1)}(0,3)} - 1 = 0.027027$

  We assume that the floating rate after 1 year will be $f^{(1)}_{2,3}$. So the cash flow that results is $+0.027027 \times F \times 1$ and discounted $+0.027027 \times F \times DF^{(1)}(0,3)$
- So the cash flow after 3 year is $-S_{0,4} \times F \times DF^{(1)}(0,3) + 0.027027 \times F \times DF^{(1)}(0,3)$

* After the fourth year we have
  - A fixed flow that we pay, so $-S_{0,4} \times F \times 1$ ($\Delta t = 1$) and discounted $-S_{0,4} \times F \times DF^{(1)}(0,4)$;
  - A floating cash flow at an UNKNOWN floating rate $f^{(1)}_{3,4}$, this unknown rate can be derived from the discount rates as explained in section 3.4.3: $\frac{1}{DF^{(1)}(0,3)}\frac{1}{DF^{(1)}(3,4)} = \frac{1}{DF^{(1)}(0,3)}$ and in a similar way we find that $f^{(1)}_{3,4}) = \frac{DF^{(1)}(0,3)}{DF^{(1)}(0,4)} - 1 = 0.0393258$

    We assume that the floating rate after 1 year will be $f^{(1)}_{2,3}$. So the cash flow that results is $+0.027027 \times F \times 1$ and discounted $+0.0393258 \times F \times DF^{(1)}(0,3)$
  - So the cash flow after 4 year is $-S_{0,4} \times F \times DF^{(1)}(0,4) + 0.0393258 \times F \times DF^{(1)}(0,3)$

Therefore the value of the swap at $t = 0$ is

$$
\begin{aligned}
V &= \\
&- S_{0,4} \times F \times 0.9800 + 0.0204082 \times F \times 0.9800 \\
&- S_{0,4} \times F \times 0.9500 + 0.0315789 \times F \times 0.9500 \\
&- S_{0,4} \times F \times 0.9250 + 0.027027 \times F \times 0.9250 \\
&- S_{0,4} \times F \times 0.8900 + 0.0393258 \times F \times 0.8900
\end{aligned}
$$

The swap rate that we are looking for is the value for $S_{0,4}$ that makes the present value of these cash flows zero (else there are riskless profits from arbitrage) So we put $V = 0$ and solve for $S_{0,4}$.

$S_{0,4} = \frac{0.0204082 \times 0.9800 + 0.0315789 \times 0.9500 + 0.027027 \times 0.9250 + 0.0393258 \times 0.8900}{0.9800 + 0.9500 + 0.9250 + 0.8900} = 0.0293725$

11.3.2 Q3.2

THIS IS A VERY TRICKY EXERCISE , SO TRY IT !!!!!!!

IRS with paid fixed rate 10% p.a. and receive of 3month LIBOR. The notional is $Q = 100,000,000$ USD. Payments are veery 3 month.

The remaining life time is 14 months. Average bid and offered rates currently being swapped for 3 month is 13% p.a. for all maturities. One month ago the 3 month LIBOR was 11.8% p.a.

All rates are compounded quarterly.

Compute the value of the swap in two different ways.

ASK QUESTION ON QUARTERLY COMPOUNDING, SWAP RATE THAT IS THE SAME FOR ALL MATURITIES IMPLIES THAT SPOT RATE ALSO HAS THAT VALUE

Dates at which cash flows occur: We have to find out the cash flows and the dates at which they occur. We have an interest rate swap with a remaining life time of 14 months and we exchange 3 month LIBOR for a fixed rate, so payments are every 3 month. This that the swap must have been initialised 1 month ago. So the swap was initiated at $t - 1/12$ and cash flows will occur at $t + 2/12, t + 5/12, t + 8/12, t + 11/12, t + 14/12$.

Discount factors: we will need the discount factors to discount the cah flows at each of these dates. We do not have the zero coupon rates, but we know that the swap rates for all maturities are $S_{0,t} = 0.13, \forall t$. We can compute the discount rates from these swap rates by taking into account that it is the rate at which the discounted cash flows for a bond sum up to the face value of the bond or

- we can use the swap rate $S_{0,2/12} = 0.13$ to find the discount factor at $t + 2/12$. Indeed, for a swap with maturity $t + 2/12$ we have:
  The first disocunt factor is for $t + 2/12$!!!!
  $$F = \overbrace{F\left((1+0.13/4)^{4\times 2/12} - 1\right) \times DF^4(0, 2/12)}^{\text{discounted interest at t=2/12}} + \overbrace{F \times DF^4(0, 2/12)}^{\text{discounted face value}},$$
  we find $DF^4(0;2/12) = \frac{1}{1+(1+0.13/4)^{8/12}-1} = 0.9789037$
- we can use the swap rate $S_{0,5/12} = 0.13$ to find the discount factor at $t + 5/12$. Indeed, for a swap with maturity $t + 5/12$ we have:

$$\begin{aligned} F &= \overbrace{F\left((1+0.13/4)^{2\times 3/12} - 1\right) \times DF^4(0, 2/12)}^{\text{discounted interest at t=2/12}} \\ &+ \overbrace{F\left((1+0.13/4)^{4\times 3/12} - 1\right) \times DF^4(0, 5/12)}^{\text{discounted interest at t=5/12}} \\ &+ \overbrace{F \times DF^4(0, 5/12)}^{\text{discounted face value}} \end{aligned}$$

Note that $4 \times 3/12 = 1$ so we have So
$1 = 0.02155097 \times DF^4(0, 2/12) + 0.0325 \times DF^4(0, 5/12) + DF^4(0, 5/12),$

we already computed that $DF^4(0,2/12) = 0.9789037$ and we find that
$DF^4(0,5/12) = \frac{1-0.02155097\times 0.9789037}{1+0.0325} = 0.9480907$

- we can use the swap rate $S_{0,8/12} = 0.13$ to find the discount factor at $t+8/12$:
$DF^4(0,8/12) = \frac{1-0.02155097\times 0.9789037-0.0325\times 0.9480907}{1+0.0325} = 0.9182477$
- we can use the swap rate $S_{0,11/12} = 0.13$ to find the discount factor at $t+11/12$:
$DF^4(0,11/12) = \frac{1-0.02155097\times 0.9789037-0.0325\times 0.9480907-0.0325\times 0.9182477}{1+0.0325} = 0.889344$
- we can use the swap rate $S_{0,14/12} = 0.13$ to find the discount factor at $t+14/12$:
$DF^4(0,14/12) = \frac{1-0.02155097\times 0.9789037-0.0325\times 0.9480907-0.0325\times 0.9182477-0.0325\times 0.889344}{1+0.0325} = 0.8613501$

Summarising:

DF(0,2/12)	0.9789037
DF(0,5/12)	0.9480907
DF(0,8/12)	0.9182477
DF(0,11/12)	0.889344
DF(0,14/12)	0.86153

forward rates: The first rate is 0.118 !, it was known at initialisation of the swap but is only paid after 3 month.

The other forward rates can be found as $\frac{1}{DF(0,5/12)} = \frac{1}{DF(0,2/12)}\frac{1}{DF(2/12,5/12)}$ or

$(1+f_{2/12,5/12}/4)^{4\times 3/12} = \frac{DF(0,2/12)}{DF(0,5/12)}$ or

$f_{2/12,5/12} = 4\frac{DF(0,2/12)}{DF(0,5/12)} - 1 = 0.1300002$

$f_{5/12,8/12} = 4\frac{DF(0,5/12)}{DF(0,8/12)} - 1 = 0.1299998$

$f_{8/12,11/12} = 4\frac{DF(0,8/12)}{DF(0,11/12)} - 1 = 0.1300001$

$f_{11/12,14/12} = 4\frac{DF(0,11/12)}{DF(0,14/12)} - 1 = 0.1300001$

Cash flows: Watch out, interest for the first period also is 2 months, the other are also three

if $m\Delta t = 1$ then interest calculations are simplified !!!: $100\times[(1+S_{-1/12,14/12})^{4\times 3/12} - 1] = 100\times S_{-1/12,14/12}/4$

t	f_{t_1,t_2}	$S_{-1/12,14/12}$	$DF(0,t_2)$	Fix leg	Floating leg
2/12	0.118	0.1	0.9789037	2.8877659	2.4472593
5/12	0.13	0.1	0.9480907	3.0812948	2.3702268
8/12	0.13	0.1	0.9182477	2.984305	2.2956193
11/12	0.13	0.1	0.889344	2.890368	2.22336
14/12	0.13	0.1	0.86153	2.7999725	2.153825

The value is this =3.153416 million EUR.

Second method: difference of two bonds

11.4 Option strategies.

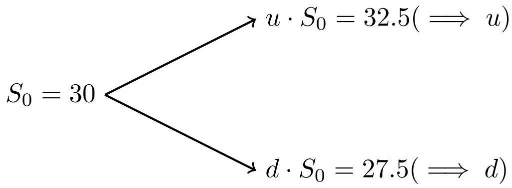
11.5 Option pricing and hedging.

11.5.1 Q5.1

$S_0 = 30$, $S_{T=3/12,u} = 32.5, S_{T=3/12,d} = 27.5$, $r = 0.08$.

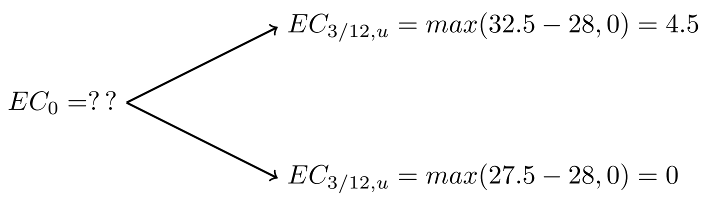
Value of a European Call on the stock, call has $K = 28$ We have no information on σ so we will have to use trees, if we know u and d. We do know these because we know $S_{T=3/12,u} = u \cdot S_0, S_{T=3/12,d} = d \cdot S_0$ so $u = \frac{32.5}{30} = 1.0833333$ and $d = \frac{27.5}{30} = 0.9166667$.

Schema for the spotprice between $t = 0$ and $t = 3/12$:



To compute the value of the call we use the schema for the European call between $t = 0$ and $t = 3/12$:

Compute backward $EC_{t-3/12} = (p \cdot EC_t^u + (1-p)EC_t^d)e^{-r \cdot 3/12}$



For computing EC_0 we have to **compute the expected value of the call at** $t = 3/12$ **using the risk neutral probability and then discount**.

The formula for the risk neutral probability is $p = \frac{e^{rT}-d}{u-d}$ and using the values supra we find that $p = 0.6212081$. p is the risk neutral probability for an upward move. As there are only two options, the risk neutral probability for a downward move is $1 - p = 0.3787919$

In order to get the expected value you multiply each outcome by its own risk neutral probability and then sum: $0.6212081 \times 4.5 + 0.3787919 \times 0 = 2.7954365$.

This is the expected value at $t = 3/12$ so after discounting we find $EC_0 = 2.7954365e^{-0.08 \times 3/12} = 2.7400831$

So $EC_0 = 2.7401$.

verify that no arbitrage and risk neutral valuation give the same result:

Risk neutral valuation is the method above.

The "no arbitrage" result is that we buy Δ shares of stock and we lend B money, where Δ and B are as in the section on the replicating portfolio. Then $EC_0 = \Delta S_0 + B$.

The formula for Δ is $\Delta = \frac{EC_u - EC_d}{uS_0 - dS_0} = \frac{4.5-0}{32.5-27.5} = 0.9$ and $B = \frac{uEC_d - dEC_u}{(u-d)e^{rT}} = \frac{1.0833333 \times 0 - 0.9166667 \times 4.5}{(1.0833333-0.9166667)e^{0.08\times 3/12}} = -24.2599278$

We find that $EC_0 = \Delta S_0 + B = 0.9 \times 30 - 24.2599278 = 2.7400722$

11.5.2 Q5.2

European put, $T = 4/12, K = 2300$, $I_0 = 2323$, $r = 0.05/annum$, $q = 0.0285/annum$, $\sigma = 0.3$, Rates are in continuous compounding.

What is the value of the European put? What if it were an American put ?

Now we have to find u and d via the volatility, i.e.

$u = e^{\sigma\sqrt{\Delta t}} = e^{0.3\sqrt{1/12}} = 1.0904632$, $d = e^{-\sigma\sqrt{\Delta t}} = 0.9170415$

We can do it in a fast way now, because we need only the stock prices at the fourth step and we can compute these directly:

There are four steps in the tree, to the highest stock price is when we go four times up, i.e. $S_0u^4 = 3284.6812501$. If we know the (risk neutral) probability to go up in one move, p, then the probability to go up four times in fot moves is $\binom{4}{4}p^4(1-p)^{4-4}$.

If we go up three times and down once, then the stock price after four moves is $S_0u^3d = 2762.3023991$. The probability of going three times up and once down is $\binom{4}{3}p^3(1-p)^{4-3}$.

The point is to find the risk neutral probability p, we have seen that, for a non-dividend paying stock it is $p = \frac{e^{r\Delta t}-d}{u-d}$ for a dividend paying stock this becomes $p = \frac{e^{(r-q)\Delta t}-d}{u-d} = \frac{e^{(0.05-0.0285)1/12}-0.9170415}{1.0904632-0.9170415} = 0.4887034$

Note that for a put the value at maturity is $max(K - S_{4/12}0, 0)$

So we find the following table:

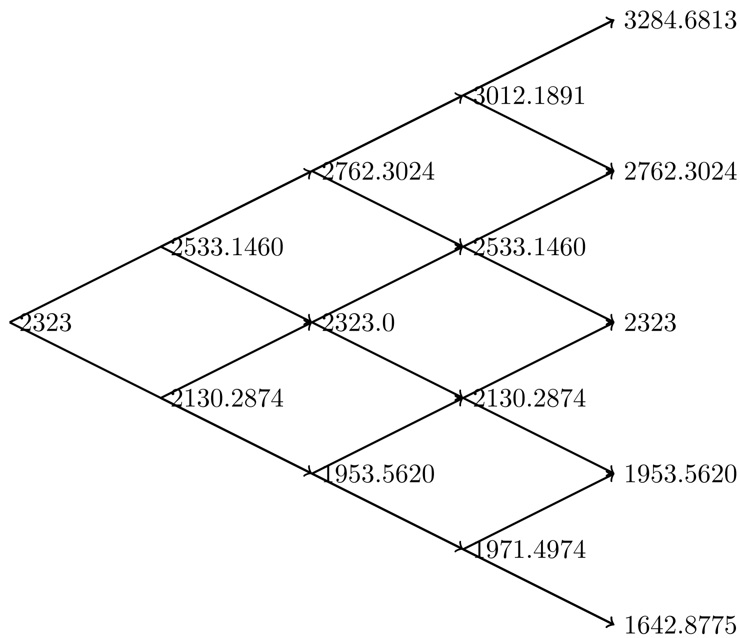
path	$S_{4/12}$	$EC_{4/12}$	probability
4 times up	$S_0u^4 = 3284.6812501$	$max(2300 - S_{4/12}, 0) = 0$	$\binom{4}{4}p^4(1-p)^{4-4} = 0.0570387$
3 up, 1 down	$S_0u^3d = 2762.3023991$	$max(2300 - S_{4/12}, 0) = 0$	$\binom{4}{3}p^3(1-p)^{4-3} = 0.2387057$
2 up, 2 down	$S_0u^2d^2 = 2323$	$max(2300 - S_{4/12}, 0) = 0$	$\binom{4}{2}p^2(1-p)^{4-2} = 0.374617$
1 up, 3 down	$S_0u^1d^3 = 1953.5620002$	$max(2300 - S_{4/12}, 0) = 346,4380$	$\binom{4}{1}p^1(1-p)^{4-1} = 0.2612942$
0 up, 4 down	$S_0u^0d^4 = 1642.8775242$	$max(2300 - S_{4/12}, 0) = 657,1225$	$\binom{4}{0}p^0(1-p)^{4-0} = 0.0683444$

Next you multiply each outcome by its probability, sum and discount so we have $(346.4380 \times 0.2612942 + 657,1225 \times 0.0683444)e^{-0.05\times 4/12} = 133.1943744$

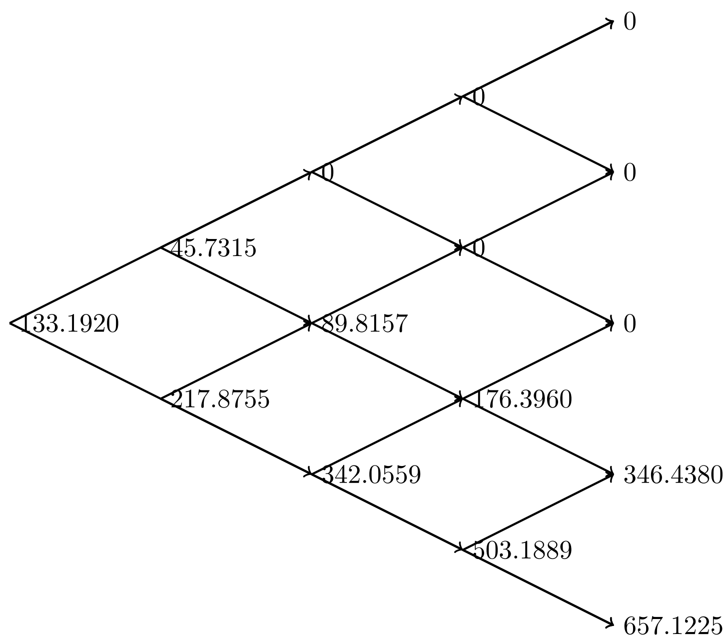
American option, how do you find that ?

To find the American option value we need to work through the whole tree, one step at a time:

First for the stock index:



For the European put option:



For an American put:

the only difference is that we can exercise an American put before $T = 4/12$, so if we keep it until $t = 4/12$ then there is no difference !

If we execute it at $t = 3/12$, then (look at the tree with the spot prices) the

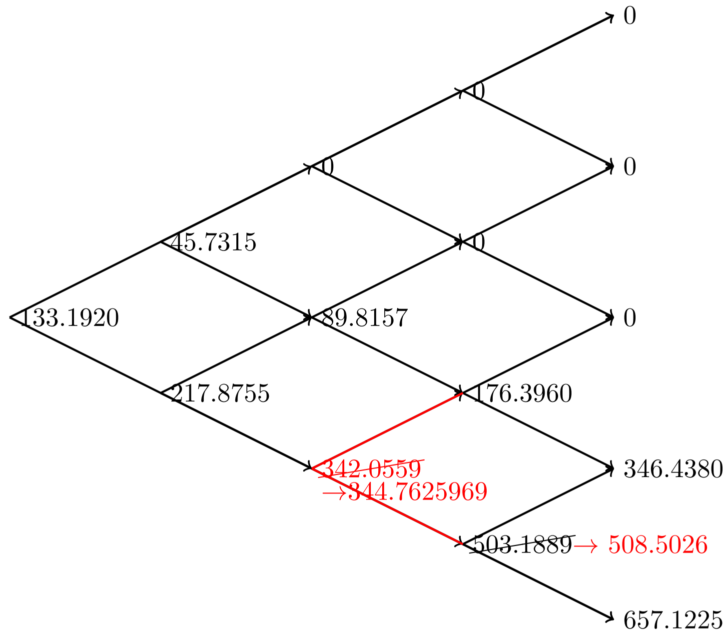
American put is worth $max(2300 - S_{3/12}, 0)$, so then the spot prices at $t = 3/12$ are from top to bottom: 3012.1891, 2533.1460, 2130.2874, 1791.4974 resp. So $max(2300 - S_{3/12}, 0)$ is resp. 0,0,0 169.7126, 508.5026.

If you compare this to the tree of the European option, then you see that there is a benefit in exercising the Amerian put for the value of 508.5026 because that yields more than keeping the put !!

But, as one nod has changed, we must re-compute the other nodes in the previous step !!!

$(0.4887 \times 176.3960 + (1 - 0.4887) \times 508.5026)e^{-r\times 1/12} = 344.7625969$

So we get the following picture



Apply the same procedure to $t = 2/12$, ...

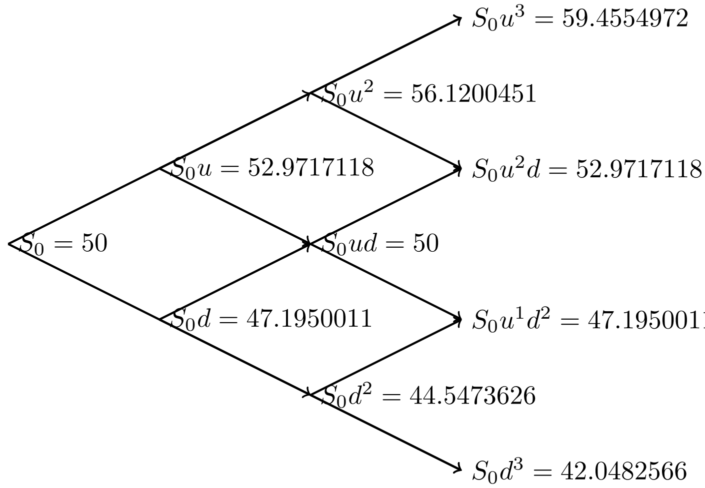
11.5.3 Q5.3

$S_0 = 50$, $\sigma = 0.2$, $r = 0.03$. $K = 55$, $T = 3/12$.

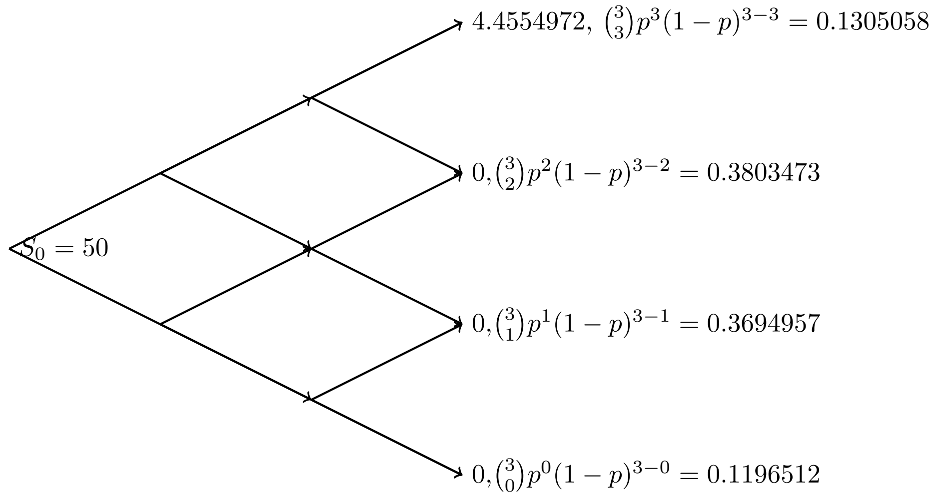
Three month call / put value, step = 1/12: $u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\sqrt{1/12}} = 1.0594342$, $d = e^{-\sigma\sqrt{\Delta t}} = e^{-0.2\sqrt{1/12}} = 0.9439$.

Risk neutral probability $p = \frac{e^{r\Delta t} - d}{u - d} = 0.5072359$

First for the <u>stock index</u>:



call: the value of a call after three steps is $max(S_T - 55, 0)$, the probabilities of the outcomes can be computed using the Binomial distribution, the schema is

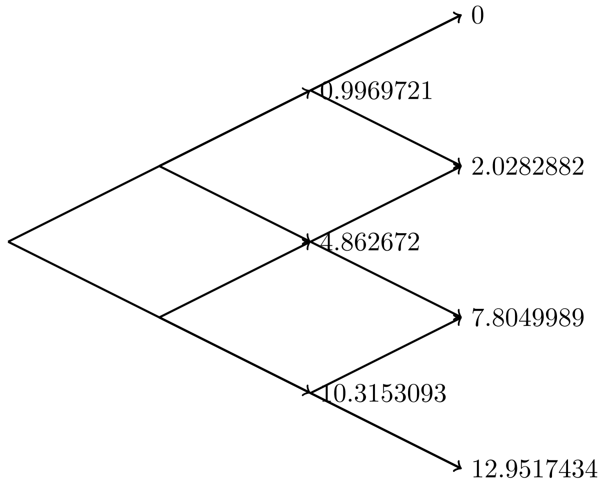


The expected payoff at $t = 3/12$ is $4.4555 \times 0.1305058 = 0.5814686$ and discounted it is $0.5814686e^{-r\times 3/12} = 0.5771239$

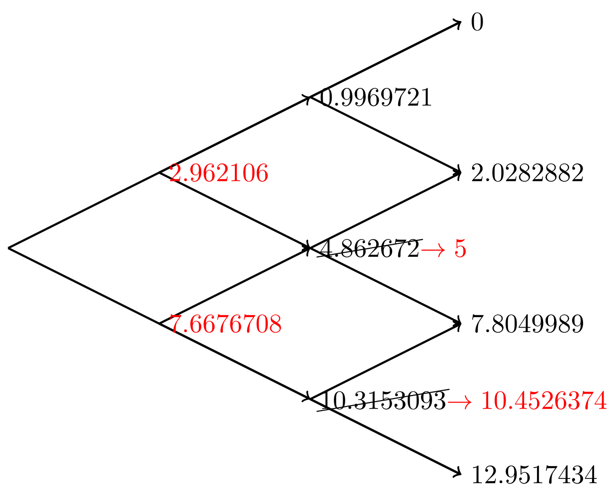
put: In a similar way you can find the value of the put.

Put-call parity: Just check the formula

The American put: Start from the diagram of the European put at the end and compute one step backwards:

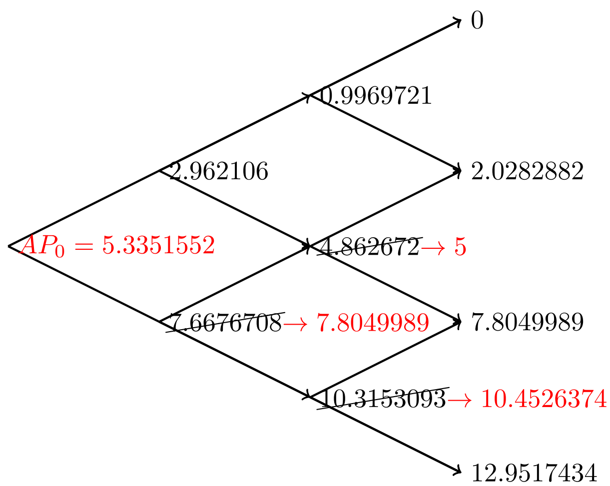


Compare this at $t = 2/12$ to the result if the American option would be executed, if favorable then change the schema: so you have to look at the tree with the spot prices at $t = 2/12$, we find (from top to bottom) 56.1200451, 50, 44.5473626. We know that we have a put with $K = 55$, so at $t = 2/12$ this would be worth $max(K - S_{2/12}, 0)$ so resp. 0, 5, 10.4526374. The values that are higher than in the tree above, for $t = 2/12$ should therefore be replaced by more favorable values in case of an early exercise of the American put:



These new values yield values at $t = 1/12$, the expected value (risk neutral) discounted so : $(0.5072359 \times 0.9969721 + (1 - 0.5072359) \times 5)e^{-0.03/12} = 2.962106$ and below $(0.5072359 \times 5 + (1 - 0.5072359) \times 10.4526374)e^{-0.03/12} = 7.6676708$

Then we do the check at $t = 1/12$, the spot prices are 52.9717118, 47.1950011 so the put with exercise price is worth 0, resp 7.8049989



Computing the last step backward we find $AP_0 = (0.5072359 \times 2.962106 + (1 - 0.5072359) \times 7.8049989)e^{-0.03/12} = 5.3351552$

With Black-Scholes: Just substitute the values in the formula of black/Scholes

11.6 Exam level questions

11.6.1 Exam Q1

We use forward exchange contracts to hedge currency risk.

- on Sept 30, 2008 we enter a forward contract to sell 10,000,000 USD for receipt of EUR on June 30 2009. How much EUR do you expect to receive ? What is the forward exchange rate ?

  Interest rates are 9-month rates in simple annual compounding: $S_0 = 1.40USD/EUR, R_{USD} = 2.00\%, R_{EUR} = 3.00\%$

  Procedure to follow for CURRENCY FORWARDS.

  1. Currency forwards are assets with a percentage profit, so we will use the formula $F_0 = S_0 e^{(r-q)T)}$, however you there are two currencies so we first must determine which ine is the asset and which one is the currency we use to pay
     - The currency that is mentioned in the qunatity-parameter Q of the contract is the asset because Q tells how many assets will be sold/bought at T. In this exercise it is $Q = 10,000,000$ USD, so the asset is 1USD.
     - The currency used to pay is the other one so it is EUR.
  2. Form this it follows what r and q are: r is the risk free rate on the currency used to pay, so $r = R^{EUR}$, q is the rate of profit on the asset, so $q = R^{USD}$;
  3. ALWAYS check the compounding rule. In this excercise it is simple annual comounding, so the formula we will have to use for the value in the contract is
  $$F_t^{s,EUR/Asset} = S_t^{EUR/Asset} \frac{1 + R_t^{EUR} \Delta t}{1 + R_t^{USD} \Delta t} \tag{30}$$
  4. for the value value of the contract is the discounted value of $F_t - F_0$ so we must know the discounting frquency and also the discount rate.
     - The discounting frequency for this excercise is simple annual accounting
     - the discount rate is the one one the currency that is used for paying, therefore R_t^{EUR}
     - The exercise says that we sell so we have a short position so we need $F_0 - F_t$ So the formula to use is:
  $$f_t^{EUR/Asset} = (F_0^{EUR/Asset} - F_t^{EUR/Asset}) \frac{1}{1 + R_t^{EUR} \Delta t} \tag{31}$$

  From the above formulas we know that $F_0^{s,EUR/USD} = S_0^{EUR/USD} \frac{1+R_0^{EUR}(T-0)}{1+R_0^{USD}(T-0)}$, where $S_0^{EUR/USD} = 1/S_0^{USD/EUR} = 1/1.4$ EUR/USD, $R_0^{EUR} = 0.03$,

$T = 9/12$, $R_0^{USD} = 0.02$, so we find that $F_0^{s,EUR/USD} = 0.7195637$ EUR/USD.

We are asked to use the same convention as in the exercise, so it should be in USD/EUR so $F_0^{s,USD/EUR} = 1.3897311$ USD/EUR.

At T we will sell 10,000,000 Assets (=USD) at 0.7195637 EUR/USD, so we will receive $7,195,636.87544$ EUR

- At the year end, December 30, 2008 se have $S_0 = 1.35$ USD/EUR, $R_{USD} = 0.5\%$, $R_{EUR} = 2\%$, what is the new forward exchange rate for delivery June 30, 2009 ? What is the value of the contract at that date ?

  From the above formulas we know that $F_t^{s,EUR/USD} = S_t^{EUR/USD} \frac{1+R_t^{EUR}(T-t)}{1+R_t^{USD}(T-t)}$, where $S_t^{EUR/USD} = 1/S_t^{USD/EUR} = 1/1.35$ EUR/USD, $R_t^{EUR} = 0.02$, $T = 9/12 - 3/12 = 6/12$, $R_t^{USD} = 0.005$.

  We find that $F_t^{s,EUR/USD} = 0.7462824$ EUR/USD. So, in the conventions of the exercise $F_t^{s,USD/EUR} = 1.3399752$ USD/EUR.

  $f_t^{EUR/Asset} = (F_0^{EUR/Asset} - F_t^{EUR/Asset}) \frac{1}{1+R_t^{EUR} 6/12} = -0.0264542$ EUR/USD.

  So we $Q = 10,000,000$ USD we have $-264,542.12375$ EUR

11.6.2 Exam Q2

I entered a forward contract to buy a share on 31/12/2009. The delivery date is 30/9/2010. After six months, the company pays a dividend. Furthermore $S_0 = 50$ EUR, $R_{9m} = 4\%$, $R_{6m} = 2\%$, $D_{6m} = 2$ EUR.

- determine the forward price.

  $F_0 = (S_0 - I_0)(1 + r\Delta t)$, where simple annual compounding is assumed.

  $I_0 = 2/(1 + 0.02 \times 6/12) = 1.980198$.

  $S_0 = 50$ EUR, $r = 0.04$, $T = 9/12$.

  So $F_0 = (50 - 2/(1 + 0.02 * 6/12)(1 + 0.04 \times 9/12) = 49.460396$ EUR.

- 31/3/2010 the stock trades at 45 EUR. The company anounced to scrap its dividend. What would be the value of the contract at 31/3/2010 ? $S_0 = 45$ EUR, $R_{6m} = 4\%$, $D_{3m} = 0$ EUR.

  , where simple annual compounding is assumed.

  $I_t = 0$.

  $S_t = 45$ EUR, $r = 0.04$, $\Delta t = 6/12$.

  So $F_t = 45(1 + 0.04 \times 6/12) = 45.9$ EUR.

  $f_f = (F_t - F_0)\frac{1}{1+r\Delta t} = -3.4905843$

11.6.3 Exam Q3

Hedge an interest rate exposure for $Q = 100,000,000$ EUR for the period 1/4/2010-30/6/2010 using a forward rate agreement (FRA) with a fixed rate of 3%. On 31/12/2009 the short-term interest rates in simple annual accounting (SAC) are:

Maturity (months)	Rate (SAC)
3	2.5%
6	3.0%
9	3.5%
12	4.0%

- Wat is the fair value of the FRA. Note we are asked for the fair value of the agreement, i.e. of the contract.

  So we need F_0, F_t, and we are said to use SAC.

  We have an interest rate in the contract, this rate was fixed at $t = 0$ so F_0 is 3%.

  At 31/12/2009 we now have to find the from 1/4/2010 until 30/6/2010, which is in the future so we look for a forward rate. As said we do not need to know the formula, we can easily derive it using the schema in figure 1. We have to use SAC so $(1 + R_{0,T}T) = (1 + R_{0,t}t)(1 + f_{t,T}(T - t))$ or

  $f_{t,T} = \frac{1}{T-t}\left(\frac{(1+R_{0,T}T)}{(1+R_{0,t}t)} - 1\right)$

  where $t = 3/12$, $T = 6/12$, $R_{0,T} = 0.03$, $R_{0,t} = 0.025$, so we find

  $f_{t,T} = 0.0347826$.

  This is not the value of the contract !!

  We now have a contract (FRA) with a rate fixed at 0.03 and the current forward rate at 31/12/2009 is $f_{t,T} = 0.0347826$.

  - With the contract you pay a rate of 0.03 on 100,000,000 EUR from 1/4/2010 until 30/6/2010, or $100000 \times 0.03 \times 3/12 = 750,000.00000$ at at T
  - With the current rate you pay a rate of $f_{t,T} = 0.0347826$ on 100,000,000 EUR from 1/4/2010 until 30/6/2010, or $100,000,000 \times 0.0347826 \times 3/12 = 869,565.21739$ at T

  So the possessor of the contract has a "rate advantage" $869,565.21739 - 750,000 = 119,565.21739$ EUR at $T = 30/6/2010$!!!!!!!.

  We are asked for the value of the contract at 31/12/2019 and we have the value at T=30/6/2010 so we have to discount this value using SAC:

  so we find that $f_t = 117,798.24373$ EUR

11.6.4 Exam Q6

Par interest rate swaps on an annual basis:

Maturity	Rate
1	2.00%
2	2.50%
3	3.00%
4	3.50%

- Use bootstrapping to derive the cash flows after 1,2,3,4 years.

  $1 = (1 + S_{0,1})DF(0,1) \implies DF(0,1) = \frac{1}{1+0.02} = 0.9803922$

  $1+S_{0,2}DF(0,1)+(1+S_{0,2})DF(0,2) \implies DF(0,2) = \frac{1-0.025\times0.9803922}{1+0.025} = 0.9516978$

  $1 + S_{0,3}DF(0,1) + S_{0,3}DF(0,2) + (1 + S_{0,3})DF(0,3) \implies DF(0,3) = \frac{1-0.03\times0.9803922-0.03\times0.9516978}{1+0.03} = 0.9145993$

- IMPORTANT: swap rates are not compounded, they are paid every period, so the forward rates can not be derived with the swap rates, you MUST used the discount factors !!

  Using the procedure schematised in figure 1, so $(1 + R_{0,1})(1 + f_{1,2}) = (1 + R_{0,2})^2$ but the table above does not give the $R_{t,T}$ but the swap rates $S_{t,T}$!!!

  The rates $R_{t,T}$ are implicit in the discount factors, se re-write this with discount factors !! $\frac{1}{DF(0,1)}(1 + f_{1,2}) = \frac{1}{DF(0,2)}$ or $(1 + f_{1,2}) = \frac{DF(0,1)}{DF(0,2)} = 1.0301507$

  We are asked for the coupon rate for a bond startinf in one year and maturing after 2 years , so we need $f_{1,3}$, similar as before we find that $f_{1,3} = \frac{DF(0,1)}{DF(0,3)} - 1 = 0.0719363$ but we need the coupon rate, $f_{1,3}$ is over two year, so we have to divide $f_{1,3}/2 = 0.0359682$ or, in case of compounding $(1 + x)^2 = 1 + f_{1,3}$ and solve for x and $x = \sqrt{1 + f_{1,3}} - 1 = 0.0353436$

  check because here I don't know, how to find the coupon rate ?????

11.6.5 Exam 7

The forward rates are given: $f_{0,1} = R_{0,1} = 0.02$, $f_{1,2} = 0.025$, $f_{2,3} = 0.03$, $f_{3,4} = 0.035$.

Discount rates: $DF(0,1) = \frac{1}{1+R_{0,1}} = 0.9803922$, $DF(0,2) = \frac{1}{1+R_{0,1}}\frac{1}{1+f_{1,2}} = 0.9564802$, $DF(0,3) = 0.9286215$, $DF(0,4) = 0.8972188$

Swap rate: The fixed rate that makes a (fix rate) bond equal to its face value so

watch out, each term is interest for one single year, also a term to discount F in last period !

$F = S_{0,4} \times F \times DF(0,1) + S_{0,4} \times F \times DF(0,2) + S_{0,4} \times F \times DF(0,3) + S_{0,4} \times F \times DF(0,4) + F \times DF(0,4)$, F is on both sides , solve for $S_{0,4}$

11.6.6 Exam 8

$DF(0,6/12) = 0.98, DF(0,9/12) = 0,965, DF(0,1) = 0.955, DF(0,2) =?, DF(0,3) = 0.85$.

Forward rate: $f_{6/12,9/12}$: follows from $\frac{1}{DF(0,9/12)} = \frac{1}{DF(0,6/12)}(1+f_{6/12,9/12})$

Find DF(0,2) : compute DF(0,2) is 2-year swap rate is 0.04.

The problem here is that we do not know the dates of the cash flows in the swap.

If (???!!!!!!) the cash flows in the swap are exchanged every year, then the swap rate is the fixed rate that makes the present value of a bnd equal to its face value:

watch out, interest every year !! and discount it

$$F = \overbrace{0.04 \times F \times DF(0,1)}^{\text{interest first}} + \overbrace{0.04 \times F \times DF(0,2)}^{\text{interest second}} + \overbrace{F \times DF(0,2)}^{\text{face after 2y}}$$

The only unknown here is $DF(0,2)$ so we can compute it !

What if cash flows are paid quarterly ??

Then $DF(0,1) = \frac{1}{(1+R_{0,1}/4)^{4\times 1}}$, or $R_{0,1} = 4 \times \sqrt[4]{\frac{1}{DF(0,1)}} - 1 = 0.04631$

In that case we would find

$DF(0,6/12) = \frac{1}{(1+0.04630996/4)^{4\times 6/12}} = 0.977241$

$DF(0,9/12) = \frac{1}{(1+0.04630996/4)^{4\times 9/12}} = 0.9660565$

These do not seem to be the same values as given supra, so I think it is not quatrely.

Fixed rate for 3y swap: we now know DF(0,1), DF(0,2), DF(0,3) so we can find $S_{0,3}$ using a bond with a fixed rate:

$$F = \overbrace{S_{0,3} \times F \times DF(0,1)}^{\text{interest first}} + \overbrace{S_{0,3} \times F \times DF(0,2)}^{\text{interest second}} + \overbrace{S_{0,3} \times F \times DF(0,3)}^{\text{interest third}} + \overbrace{F \times DF(0,3)}^{\text{face after 3y}}$$

The F is on both sides so can be dropped, DF(0,1), DF(0,2) and DF(0,3) are known, so this can be solved for $S_{0,3}$.

11.6.7 Exam 9

THE SOLUTION IN HIS SLIDES IS EASIER, BUT BE SURE YOU UNDERSTAND THE START FORMULA

see forward starting swap for an explanation !

MAKE THIS ONE, IT COMBINES ALL THE ELEMENTS !!!

$R_{0,6/12} = 0.014, S_{0,1} = 0.017, f_{12/12,18/12} = 0.018, S_{0,2} = 0.02, f_{24/12,30/12} = 0.025$.

t	fix	float	DF	given
0				
6/12	start swap	start swap		$R_{0,6/12} = 0.014$
12/12		$f_{6/12,12/12} \times F \times 6/12$	DF(0,12/12)	$S_{0,1} = 0.017$
18/12	$-S \times F \times 12/12$	$f_{12/12,18/12} \times F \times 6/12$	DF(0,18/12)	$f_{12/12,18/12} = 0.018$
24/12		$f_{18/12,24/12} \times F \times 6/12$	DF(0,24/12)	$S_{0,2} = 0.02$
30/12	$-S \times F \times 12/12$	$f_{24/12,30/12} \times F \times 6/12$	DF(0,30/12)	$f_{24/12,30/12} = 0.025$

I AM NOT SURE ABOUT THIS? SO PAY ATTENTION IN THE LESSON:

I would say that the present value of all these flows must be zero, because else there are arbitrage possibilities. So,

$$\begin{aligned} 0 &= f_{6/12,12/12} \times \cancel{F} \times 6/12 \times DF(0,12/12) \\ &+ (-S \times \cancel{F} \times 12/12 + f_{12/12,18/12} \times \cancel{F} \times 6/12) \times DF(0,18/12) \\ &+ f_{18/12,24/12} \times \cancel{F} \times 6/12 \times DF(0,24/12) \\ &+ (-S \times \cancel{F} \times 12/12 + f_{24/12,30/12} \times \cancel{F} \times 6/12) \times DF(0,30/12) \end{aligned}$$

$$\begin{aligned} 0 &= f_{6/12,12/12} \times DF(0,12/12) \\ &- 2S \times DF(0,18/12) + f_{12/12,18/12} \times DF(0,18/12) \\ &+ f_{18/12,24/12} \times DF(0,24/12) \\ &- 2S \times DF(0,30/12) + f_{24/12,30/12} \times DF(0,30/12) \end{aligned}$$

$2S \times DF(0,30/12) + 2S \times DF(0,18/12) = f_{6/12,12/12} \times DF(0,12/12) + f_{12/12,18/12} \times DF(0,18/12) + f_{18/12,24/12} \times DF(0,24/12) + f_{24/12,30/12} \times DF(0,30/12)$ or

$$S = \frac{f_{0.5,1} \times DF(0,1) + f_{1,1.5} \times DF(0,1.5) + f_{1.5,2} \times DF(0,2) + f_{2,30/12} \times DF(0,30/12)}{2(DF(0,30/12) + DF(0,1.5))}$$

Now the problem is reduced to finding all the discount factors (and with these we can compute all the forward rates). As we have information (see column "given" in the table) at every $t+i$ that we have to find a discount factor for, we will be able to find that:

- DF(0,6/12) can be found from $DF(0,6/12) = \frac{1}{1+R_{0,6/12} \times 6/12}$
- with $S_{0,1}$ you can find $DF(0,1 = 12/12)$ using the fact that $S_{0,1}$ is the fixed rate that makes a bond havinf its face value, i.e. $F = F \times S_{0,1} \times 1 \times DF(0,1) + F \times DF(0,1)$ from which you find $DF(0,1)$
- DF(0,18/12) can be found from $DF(0,18/12) = DF(0,12/12) \times \frac{1}{(1+f_{12/12,18/12})}$

- DF(0,2) can be found from $S_{0,2}$ similar as we did for DF(0,1)
- DF(0,30/12) can be found similar as we did for DF(0,18/12)

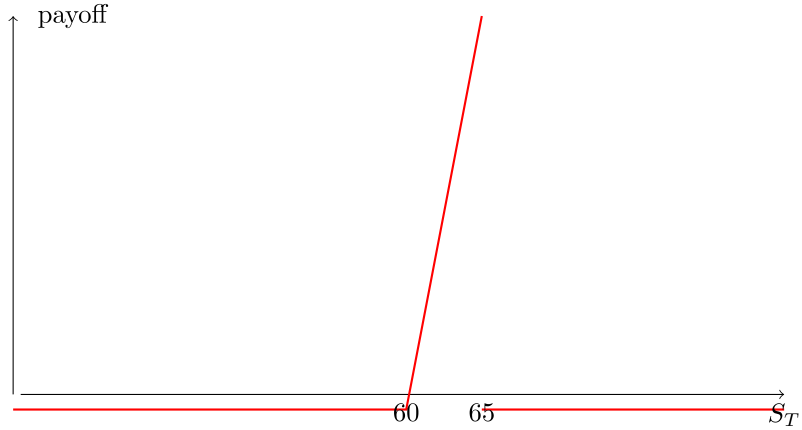
With these DF-values we can compute the f_{t_1,t_2} and then solve the whole equation for S

11.6.8 Exam 13

long call option, K= 60 EUR, M=3/12, knock-out $\geq$ 65 EUR, $S_0 = 58$ EUR, $\sigma = 0.15$, no dividends, $r = 0.04$ continuous compounding

METHOD 1 : backward computation

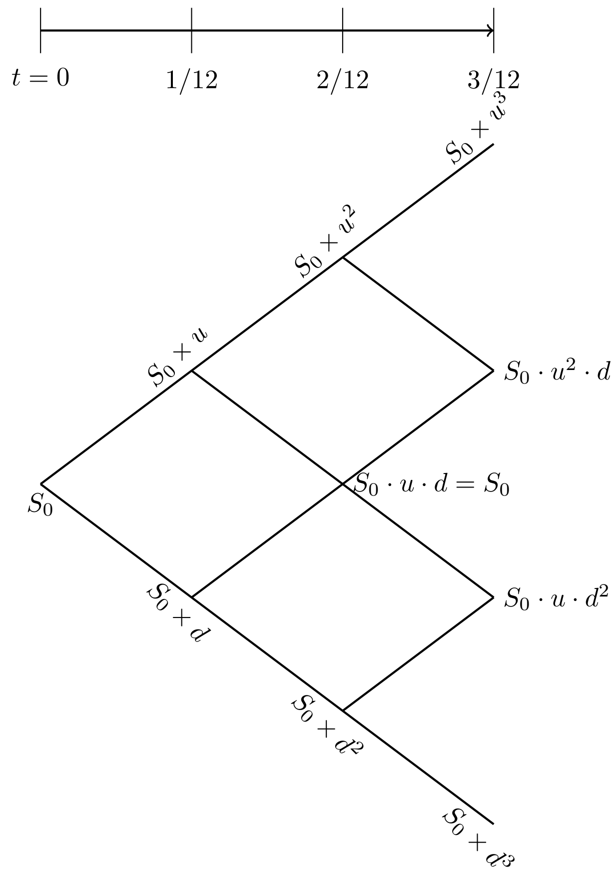
Payoff diagram: call option , but falling to zero at 65 ?



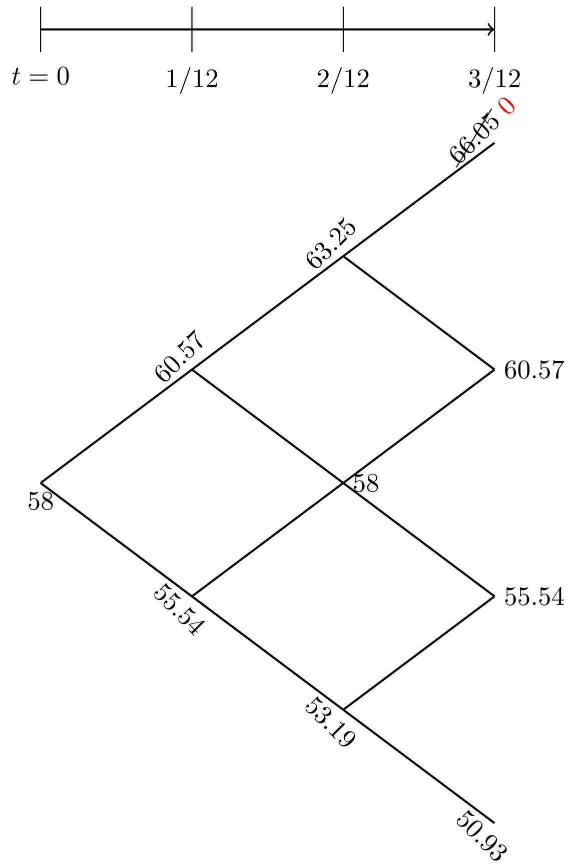
Decompose into option strategies:

Binomial tree , $\Delta t = 1/12$.

$u = e^{\sigma\sqrt{\Delta t}} = e^{0.15\times\sqrt{1/12}} = 1.0442524$
$d = e^{\sigma\sqrt{\Delta t}} = e^{-0.15\times\sqrt{1/12}} = 0.9576228$
$p = \frac{e^{r\Delta t}-d}{u-d}$, $p = 0.5277191$ is the risk neutral probability.
Note that we are not asked to compute q, if that would have been the case we would have needed information on the μ !!

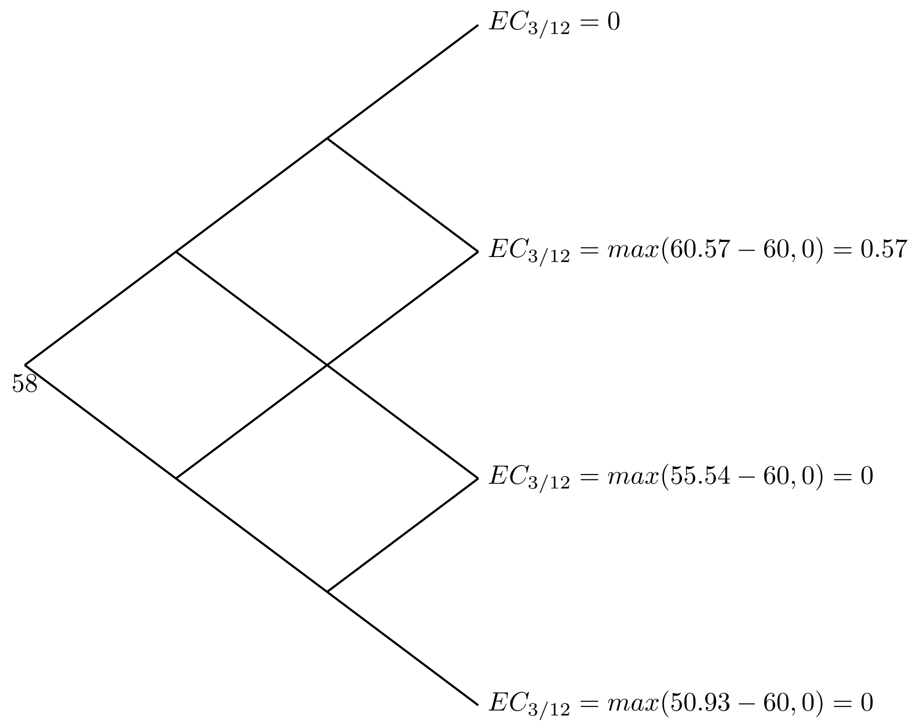

$t = 0$
$1/12$
$2/12$
$3/12$
$S_0 + u^3$
$S_0 + u^2$
$S_0 + u$
$S_0 \cdot u^2 \cdot d$
S_0
$S_0 \cdot u \cdot d = S_0$
$S_0 + d$
$S_0 \cdot u \cdot d^2$
$S_0 + d^2$
$S_0 + d^3$

Substituting the values for u and d we find the prices in the figure below:



WHEN ABOVE 65 IT DROPS TO ZERO !!!!!!! IK BEN NIET ZEKER HIER DUS LUISTER IN DE LES !!!!!!

The maturity is 3 months, so at the right hand side we have the values when the option ends, so there the value of the European call is $max(S_T - K, 0)$, as $K = 60$ we



The rest is extremely easy, you go back one period, and at each intersecting node you use the risk neutral probability to compute the expected value in that node and discount it to the right period. e.g. for month 2/3, the highest node (and knowing that p=0.4891769) we find

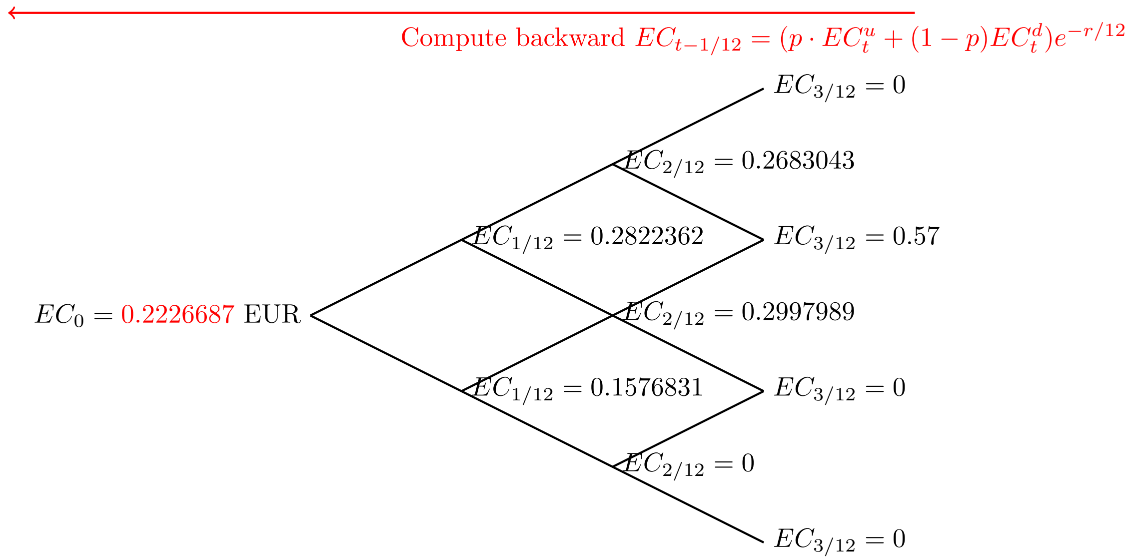
NOT SURE HERE because of the 0 with tke knockout. SO PY ATTENTION IN THE COURSE

$EC_{2/12} = (0.5277191 \times 0 + (1 - 0.5277191) \times 0.57)e^{-0.04\times 1/12} = 0.2997989$

The node below has the value $(0.5277191 \times 0.57 \times 0.57)e^{-0.04\times 1/12} = 0.2683043$

The other $EC_{2/12}$ are similar and they are zero.

Then from $EC_{2/12}$ you move back to $EC_{1/12}$ ...



The answer is $EC_0 = 0.22$ EUR.

HIJ ZOU HIER OOK KUNNEN VRAGEN NAAR Δ en/of naar q

IK WEET NIET HOE HET VOOR EEN PUT GAAT, CALL-PUT parity ??

METHOD 2 : Binomial probabilities

You have to know all possible values for the option at maturity, and their probabilities (note that $p = 0.5277191$ and $\binom{N}{n} = \frac{N!}{n!(N-n)!}$):

t	value	probability
3/12	0	$\binom{3}{3}p^3(1-p)^0 = 0.1469631$
3/12	0.57	$\binom{3}{2}p^2(1-p)^1 = 0.3945729$
3/12	0	$\binom{3}{1}p^1(1-p)^2 = 0.353122$
3/12	0	$\binom{3}{0}p^0(1-p)^3 = 0.1053419$

We have to multiply each value with its probability and sum to get the expected value at maturity. The values equal to zero will obviously not contribute so you find for the expected value at maturity $\mathbb{E}(EC_{3m}) = 0 \times 0.1469631 + 0.57 \times 0.3945729 + 0 \times 0.353122 + 0 \times 0.1053419 = 0.57 \times 0.3945729 = 0.2249066$.

Discounting at the risk free rate $r = 0.04$ with $\Delta t = 3/12$ gives for the value today $EC_0 = 0.2249066e^{-0.04 \times 3/12} = 0.2226687$ which is the same value as the one we found with backward computation.

The answer is $EC_0 = 0.22$ EUR.

11.6.9 Exam 14

Call option on a future, $T = 3/12$, $K = 60$. The future price today is $F_0 = 58$, $r = 0.01$, $\sigma = 0.5$.

Use a hedging portfolio to find EC_0: This is explained in section 7.4. We have to consturct a portfolio of a loan B and sell Δ futures, such that this portfolio has the same outcome as our option in $t = 3/12$.

$u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$ are computed in the usual way, $F_u = F_0 u$, $F_d = F_0 d$. $EC_u = max(F_u - K, 0), EC_d = max(F_d - K, 0)$.

The risk neutral probability is $\tilde{p} = \frac{1-d}{u-d}$.

Compute the value using a Binomial tree of three steps: This is similar to Binomial trees we already handled, but with another definition of EC_u, EC_d and the risk neutral probability.

Delta hedge: The number of underlyings we have to have in our portfolio, or Δ according to the formula in section 7.4.

11.6.10 Exam 15

(See Hull example 18.2.)

Put option, European

You sell the asset "EUR", so the underlying is A = 1 EUR, the stock price for A is $S_0 = 1.3$ USD/A the strike price in the contract is $K = 1.4$ USD/A. The money in which you pay the asset is USD.

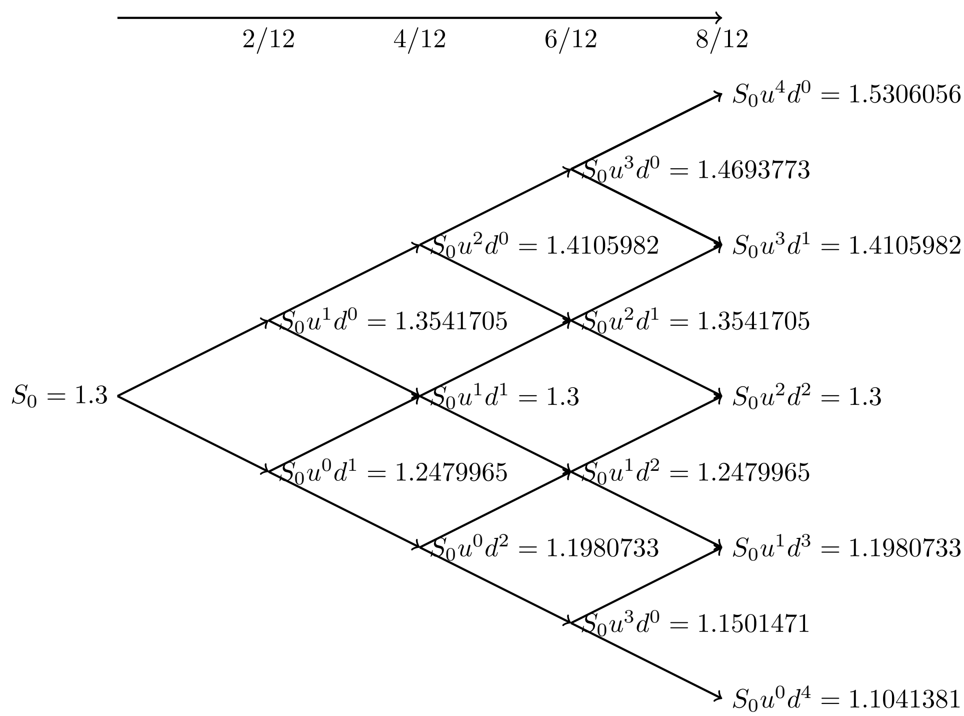
r is the interest rate in the money you pay with, so $r = 0.03$ and it is like a stock option with dividend yield because there is interest, the divident is on the underlying asset A (EUR) so $q = 0.02$.

The volatility is $\sigma = 0.1$.

so $u = e^{\sigma\sqrt{\delta t}} = e^{0.1\times\sqrt{2/12}} = 1.0416696$, $d = 0.9599973$

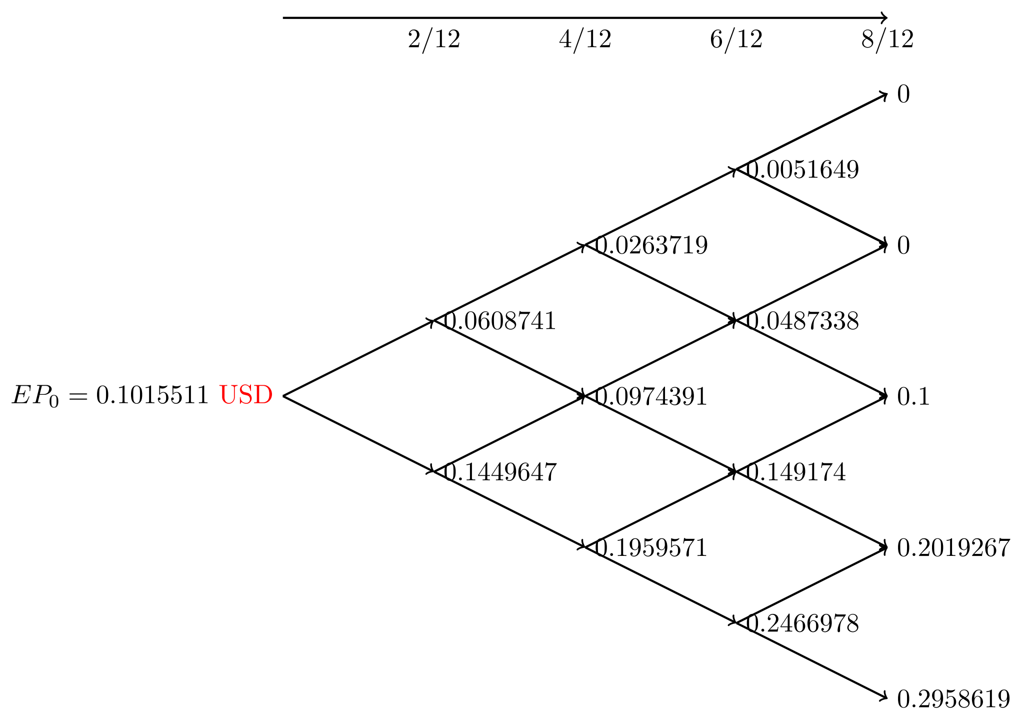
The risk neutral probability is for an option with dividend yield so $p = \frac{e^{(r-q)\Delta t} - d}{u-d} = 0.510219$

Construct the tree for the spot exchange rate: The tree is as below:



Tree for the option, unit ? The put option will be exercised at $T = 8/12$ whenever the strike price K is above teh spot price at T because then the price in the contract is advantageous to the owner of the put (he can sell at K while at the spot he gets less than K.). This gives the values at $T = 8/12$ in the tree below:

The values at $t = 3/12$ can be found using the risk neutral probability and the discount factor e.g. $(0.510219 \times 0.1306056 + (1 - 0.510219) \times 0.0105982)e^{-0.03\times 2/12} = 0.07147$



The asset is EUR and the money with which you pay is USD, so it is USD.

If it were an American option, would it be advantageous to exercise at node 0.2467 ?

If we would do that , then (see node in spot price tree) it would be worth $1.4 - 1.1501471 = 0.2498529$ If we wait (as in the European put tree) then the value is 0.2467 which is less, so it is profitable to execute the American option at that node.